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Isotopy and geotopy for ternary rings of projective planes ☆

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Abstract

We investigate the question of when two ternary rings coordinatize the same finite projective plane. Necessary and sufficient conditions are obtained for right quasifields and for right distributive linear rings whose multiplicative loop is a group.

Keywords: Projective planes; Ternary rings; Isotopy

Let P be a projective plane. By a result of M. Hall (cf. Chapter 20 in [H] or Chapter V in [HP]), given an ordered triple $\Delta = (p_1, p_2, p_3)$ of noncollinear points of P, and a point p_4 collinear with none of the three lines $p_i + p_j$, $1 \leqslant i < j \leqslant 3$, P is coordinatized by a so-called planar ternary ring $X(\mathcal{B}) = (R, T)$ with respect to the *basis* $\mathcal{B} = (\Delta, p_4)$, and (R, T) has an associated loop ring $R = (R, +, \cdot)$. Moreover, bases $\mathcal{B}$ and $\bar{\mathcal{B}} = (\bar{\Delta}, \bar{p}_4)$ are conjugate under $Aut(P)$ if and only if $X(\mathcal{B})$ is isomorphic to $X(\bar{\mathcal{B}})$, and the *triangles* Δ and $\bar{\Delta}$ are conjugate if and only if $X(\mathcal{B})$ is isotopic to $X(\bar{\mathcal{B}})$. (See Sections 1–4 for definitions of terminology, and Section 4 for proofs of the final two well-known results.)

In [A1] we began the study of finite planes P linearly coordinatized by right distributive loop rings R. We called such planes *distributive linear planes*. The analysis in [A1] proceeds via the

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study of the enveloping group $G(R)$ of R. We recall two questions raised in [A1], which give some indication of the direction we feel an investigation of distributive linear planes might head:

Question 1. Let R be a finite right distributive planar loop ring. Is it true that either

(1) R is a right quasifield, or
(2) the multiplicative loop $(R^{\#}, \cdot)$ of R is a group, and the plane $\mathcal{P}(R)$ linearly coordinatized by R is a translation plane which is also linearly coordinatized by a nonassociative left nearfield N with the same multiplicative loop as R.

Recall a *right quasifield* is a right distributive loop ring R such that the additive loop $(R, +)$ of R is a group, and a quasifield is a *nearfield* if its multiplicative loop is a group. In the second case of Question 1, the coordinatizations of $\mathcal{P}(R)$ by R and N are determined by nonconjugate triangles.

A weaker version of Question 1 asks:

Question 3. If R is a finite right distributive planar loop ring, is it true that either

(1) R is a right quasifield, so $(R, +)$ is a group, or
(2) $(R^{\#}, \cdot)$ is a group.

Define two ternary rings $X = (R, T)$ and $\bar{X} = (\bar{R}, \bar{T})$ to be *geotopic* if the plane $\mathcal{P}(X)$ coordinatized by X is isomorphic to $\mathcal{P}(\bar{X})$. From the discussion above, if X is isotopic to $\bar{X}$ then X and $\bar{X}$ are geotopic, but almost always the converse is false.

In this paper we begin to investigate geotopy of finite ternary rings, with the emphasis on the two cases that Question 3 suggests might be of most interest: Geotopy among linear rings R in $\mathbf{R}^q$ or $\mathbf{R}^!$, where $\mathbf{R}^q$ is the class of finite right quasifields, and $\mathbf{R}^!$ is the class of finite right distributive planer loop rings R whose multiplicative loop is a group.

We begin by carefully writing down some formalism for studying isotopy and geotopy of ternary rings. Much of this is well known, but the author is unaware of a reference which records and proves the necessary results, although this may just reflect the author's ignorance of the literature. Once these preliminaries are in place, we go on to use them to establish some deeper results:

First, we prove in 10.7 that if N is a finite left nearfield which is not a field, and $P = \mathcal{P}(N)$ is the plane coordinatized by N at the triangle $\Delta = (p_1, p_2, p_3)$, then P is linearly coordinatized at the triangle $\bar{\Delta} = (p_1, p_3, p_2)$ by a ring $R \in \mathbf{R}^!$, which is not a right quasifield, but which has the same multiplicative loop as N. This fact appears to have been known to a few specialists (although perhaps from a slightly different perspective), such as Kantor and Yaqub; see, for example, Kantor's remark on page 344 of [K]. In any event this observation about rings geotopic to left nearfields, shows that the following naive version of Question 1 has a negative answer: Is every finite right distributive planer loop ring a right quasifield?

We also show in 6.3 and 10.8, that a geotopy class of ternary rings containing $R \in \mathbf{R}^!$ with $R^{\#}$ nonabelian, contains at most two isotopy classes from $\mathbf{R}^!$: the class of R and the reflection of that class, as defined below. We ask in [A2]: If $R \in \mathbf{R}^!$ with $R^{\#}$ abelian, is R a field, so that $\mathcal{P}(R)$ is Desarguesian?

Perhaps most important, we investigate and (in some sense) answer one of the fundamental questions in the theory of finite projective planes: When are two finite right quasifields geotopic? To precisely state our results, one needs some familarity with various notions in Sections 6 through 8. Thus in this introduction, we restrict ourselves to the following somewhat imprecise discussion:

Let R and $\bar{R}$ be geotopic right quasifields. Then R and $\bar{R}$ coordinatize the same plane P, from the point of view of triangles $\Delta = (p_1, p_2, p_3)$ and $\bar{\Delta} = (\bar{p}_1, \bar{p}_2, \bar{p}_3)$, respectively. If P is Desarguesian then R and $\bar{R}$ are isomorphic fields, so we assume P is not Desarguesian. Then $p_3 = \bar{p}_3$, and conjugating in $Aut(P)$, we may assume $p_1 + p_2 = l(0,0) = \bar{p}_1 + \bar{p}_2$. We show that R and $\bar{R}$ are geotopic iff they are equivalent via an equivalence relation $\simeq$ which is the transitive extension of two basic types of moves: reflections and shifts. Here $\bar{R}$ is a *reflection* of R if $\bar{\Delta} = (p_2, p_1, p_3)$, while $\bar{R}$ is a *shift* of R to $x \in R^{\#}$ if $\bar{\Delta} = (\bar{p}_1, p_2, p_3)$ with $p_1 = p(0,0)$ and $\bar{p}_1 = p(0,x)$. Our theory is built around the enveloping groups $G(R)$ and $G(\bar{R})$, or more precisely the Hall systems of R and $\bar{R}$. In particular, if K_L and $K_{\bar{L}}$ are the sets of multiplicative right translations of R and $\bar{R}$, respectively, then we prove $\bar{R}$ is a reflection of R precisely when $K_{\bar{L}} = K_L^{-1} = \{k^{-1}\colon k \in K_L\}$, while $\bar{R}$ is a shift of R to x when $K_{\bar{L}} = \{1\} \cup \{1 - R_{\cdot}(x)^{-1}k\colon k \in K_L - \{R_{\cdot}(x)\}\}$, where $1 - j$ is the difference of 1 and j in the endomorphism algebra $End(R,+)$. Moreover, there are various invariants of R which remain unchanged, or only slightly perturbed within equivalences classes of $\simeq$.

In a later paper, we introduce a partition of finite right quasifields R according to the structure of $G(R)$, determine the quasifields arising in some blocks of the partition, and use the results on geotopy established here to determine when the corresponding projective planes are isomorphic.

1. Isotopy of loops

In this section we record some facts about loops and particularly about loop isotopy, that we will need in this paper. See [Br], [Ki], or [N] for more details and proofs.

Let $X = (X, \circ)$ and $Y = (Y, *)$ be loops. For $x \in X$, $R_\circ(x)$ denotes the *right translation* $y \mapsto y \circ x$, regarded as a permutation of X. From Section 1 in [A2], the *envelope* of a loop X is $\epsilon(X) = (G, H, K)$, where $K = \{R_\circ(x)\colon x \in X\}$, G is the subgroup $Env(X)$ of $Sym(X)$ generated X, and $H = G_1$ is the stabilizer in G of the identity 1 of X. We call G the *enveloping group* of X.

An *isotopism* from X to Y is a triple $\psi = (\alpha, \beta, \gamma)$ such that α, β, and γ are bijections from X to Y, and such that for all $a, b \in X$,

$$(a \circ b)\gamma = a\alpha * b\beta.$$

The isotopism ψ is *principal* if $\gamma = 1$, and an isomorphism if $\alpha = \beta = \gamma$.

Given $a, b \in X$, define $X^{a,b}$ to be $(X, \circ_{a,b})$, where

$$x \circ_{a,b} y = \left(x R_\circ(b)^{-1}\right) \circ \left(y L_\circ(a)^{-1}\right).$$

Then

1.1.

(1) *$X^{a,b}$ is a loop with identity $a \circ b$, called a principal isotope of X.*
(2) *The map $(R_\circ(b)^{-1}, L_\circ(a)^{-1}, 1)$ is a principal isotopism from $X^{a,b}$ to X.*
(3) *Every isotopism ψ is the composition $\psi = \psi_1\psi_2$ where ψ_1 is a principal isotopism and ψ_2 is an isomorphism. In particular, each loop isotopic to X is isomorphic to some principal isotope of X.*

(4) *Let* $\psi = (\alpha, \beta, \gamma): Y \to X$ *be an isotopism, and set* $a = 1\alpha$ *and* $b = 1\beta$. *Then* $\alpha = \gamma R_\circ(b)^{-1}$ *and* $\beta = \gamma L_\circ(a)^{-1}$.
(5) *Let* $\epsilon(X) = (G, H, K)$. *Then* $\epsilon(X^{a,b}) = (G, H^{R_\circ(a)R_\circ(b)}, R_\circ(b)^{-1}K)$.

Proof. See any of the references above for (1)–(3); e.g. Chapter III, Section 1 of [Br]. See 4.2 in [Ki] for (4) and 5.5 in [N] for (5). □

We can compose isotopisms in the obvious way, and so obtain the category of loops and isotopisms, in which all morphisms are invertible.

2. Loop rings

A *loop ring* is a triple $R = (R, +, \cdot)$, where R is a set and $+$ and $\cdot$ are binary operations on R such that $(R, +)$ is a loop with identity 0, $R^\# = R - \{0\}$ is a loop under $\cdot$ with identity 1, and $x \cdot 0 = 0 \cdot x = 0$ for all $x \in R$. The loop ring R is *right distributive* if for all $x, y, z \in R$, $(x + y)z = xz + yz$.

Let $R = (R, +, \cdot)$ be a loop ring, $\epsilon(R, +) = (G_+, G_{+,0}, K)$, and $\epsilon(R^\#, \cdot) = (L, L_1, K_L)$. Thus $G_+ = Env(R, +)$ and $L = Env(R^\#, \cdot)$ are the enveloping groups of the additive and multiplicative loops of R, respectively. Define the *enveloping group* of R to be $G(R) = \langle G_+, L\rangle$, regarded as a subgroup of $Sym(R)$. The *Hall system* of R is $\mathcal{H}(R) = (R, 0, 1, K, K_L)$.

3. Isotopy of ternary rings

Let $\mathbf{R}$ be the category of loop rings and isomorphisms, and $\mathbf{T}$ the category of ternary rings and isomorphisms. That is the objects in $\mathbf{T}$ are pairs (R, T) where R is a loop ring and $T: R \times R \times R \to R$ is a 3-ary operation such that:

(T1) For all $x, y \in R$, $xy = T(x, y, 0)$.
(T2) For all $y, z \in R$, $y + z = T(1, y, z)$.

Example 3.1. For $R \in \mathbf{R}$, the *linear ternary operation* on R is the map $T_R: (x, y, z) \mapsto xy + z$.

An *isotopism* $\psi: (R, T) \to (R', T')$ of ternary rings is a triple $\psi = (\alpha, \beta, \gamma)$ of bijections from R to R' such that $\delta: 0 \mapsto 0$ for each $\delta \in \{\alpha, \beta, \gamma\}$, and for all $x, y, z \in R$,

$$T(x, y, z)\gamma = T'(x\alpha, y\beta, z\gamma).$$

An *isotopism* $\psi: R \to R'$ of loop rings is an isotopism from (R, T_R) to $(R', T_{R'})$.

For $\mathbf{X} = \mathbf{R}$ or $\mathbf{T}$ and $X, X' \in \mathbf{X}$, define $\operatorname{top}(X, X')$ to be the set of isotopisms from X to X'. If $\psi = (\alpha, \beta, \gamma) \in \operatorname{top}(X, X')$ and $\bar\psi = (\bar\alpha, \bar\beta, \bar\gamma) \in \operatorname{top}(X', \bar X)$, define $\psi\bar\psi = (\alpha\bar\alpha, \beta\bar\beta, \gamma\bar\gamma)$ to be the *composition* of ψ and $\bar\psi$.

3.2. *Let* $\mathbf{X} = \mathbf{R}$ *or* $\mathbf{T}$, $X = (R, T)$ *and* $X' = (R', T')$ *in* $\mathbf{X}$, *and assume* $\psi = (\alpha, \beta, \gamma) \in \operatorname{top}(X, X')$. *Then*

(1) $\mathbf{X}^{\mathrm{top}}$ *is a category with objects* $\mathbf{X}$ *and morphisms* $\operatorname{top}(X, \bar X)$ *from* X *to* $\bar X$.

(2) *$\psi^{-1} = (\alpha^{-1}, \beta^{-1}, \gamma^{-1}) \in \mathrm{top}(X', X)$ is an inverse for ψ in $\mathbf{X}^{\mathrm{top}}$, so all morphisms in $\mathbf{X}^{\mathrm{top}}$ are isomorphisms.*
(3) *ψ is an isotopism of loops from $(R^{\#}, \cdot)$ to $(R'^{\#}, \cdot)$.*
(4) *If T' is linear then so is T.*

Proof. Parts (1) and (2) are straightforward. Under the hypotheses of (3), for $x, y \in R$,

$$(xy)\gamma = T(x, y, 0)\gamma = T'(x\alpha, y\beta, 0\gamma) = T'(x\alpha, y\beta, 0) = x\alpha \cdot y\beta,$$

which (from Section 1) is the condition for ψ to be an isotopism of the multiplicative loops. Thus (3) holds.

Assume T' is linear, and write $*$ for the multiplication in R'. We can write $\psi = \psi_1\psi_2$ as a composition of isotopisms, where $\psi_1 = (\gamma, \gamma, \gamma)$ is an isomorphism and $\psi_2 = (\gamma^{-1}\alpha, \gamma^{-1}\beta, 1)$ is a principal isotopism. Then $X\psi_1$ is linear iff X is linear, so replacing X, X', ψ by $X\psi_1, X', \psi_2$, we may assume ψ is principal. Therefore

$$T(x, y, z) = T'(x\alpha, y\beta, z) = x\alpha * y\beta + z. \qquad (*)$$

From (2), $\psi^{-1} = (\alpha^{-1}, \beta^{-1}, 1) \in \mathrm{top}(X', X)$ is an inverse for ψ, and hence from (3), $\psi^{-1} : (R'^{\#}, *) \to (R^{\#}, \cdot)$ is a principal isotopism of loops. Let e be the multiplicative identity of R', $a = e\alpha^{-1}$, and $b = e\beta^{-1}$. Then by 1.1(4), $\alpha^{-1} = R.(b)^{-1}$ and $\beta^{-1} = L.(a)^{-1}$, while by 1.1(2), $(R'^{\#}, *) = (R^{\#}, \cdot)^{a,b}$. Therefore

$$x\alpha * y\beta = xR.(b) * yL.(a) = xR.(b)R.(b)^{-1} \cdot yL.(a)L.(a)^{-1} = x \cdot y,$$

which together with $(*)$ says T is linear, establishing (4). $\square$

3.3.

(1) *Assume $R, R' \in \mathbf{R}$ and $\psi = (\alpha, \beta, \gamma) \in \mathrm{top}(R, R')$ is a morphism in $\mathbf{R}^{\mathrm{top}}$. Then $\gamma : (R, +) \to (R', +)$ is an isomorphism of loops.*
(2) *Assume $R, \bar{R} \in \mathbf{R}$ and $\bar{\psi} = (\bar{\alpha}, \bar{\beta}, \bar{\gamma})$ is an isotopism of loops from $(R^{\#}, \cdot)$ to $(\bar{R}^{\#}, \cdot)$, such that $\bar{\gamma} : (R, +) \to (\bar{R}, +)$ is an isomorphism of loops. Then $\bar{\psi} \in \mathrm{top}(R, \bar{R})$.*

Proof. Assume the hypothesis of (1) and let $y, z \in R$. By 3.2(3),

$$(y + z)\gamma = T_R(1, y, z)\gamma = T_{R'}(1\alpha, y\beta, z\gamma) = 1\alpha \cdot y\beta + z\gamma = (1 \cdot y)\gamma + z\gamma = y\gamma + z\gamma,$$

establishing (1).

Assume the hypothesis of (2). Then

$$T_{\bar{R}}(x\bar{\alpha}, y\bar{\beta}, z\bar{\gamma}) = (x\bar{\alpha})(y\bar{\beta}) + z\bar{\gamma} = (xy)\bar{\gamma} + z\bar{\gamma} = (xy + z)\bar{\gamma} = T_R(x, y, z)\bar{\gamma},$$

establishing (2). $\square$

Given $R = (R, +, \cdot) \in \mathbf{R}$ and $a, b \in R$, define $R^{a,b}$ to be $(R, +, \circ_{a,b})$, where

$$x \circ_{a,b} y = \big(xR.(b)^{-1}\big) \cdot \big(yL.(a)^{-1}\big).$$

3.4.

(1) *$R^{a,b} \in \mathbf{R}$ has multiplicative identity $a \cdot b$, and $R^{a,b}$ is called a principal isotope of R.*
(2) *The map $(R.(b)^{-1}, L.(a)^{-1}, 1)$ is a principal isotopism from $R^{a,b}$ to R.*
(3) *For $R' \in \mathbf{R}$, each $\psi \in \mathrm{top}(R', R)$ is the composition $\psi = \psi_1\psi_2$ where $\psi_1 : R' \to R^{a,b}$ is an isomorphism from R' to $R^{a,b}$ for some principal isotope $R^{a,b}$, and ψ_2 is a principal isotopism as in* (2).
(4) *Let $\mathcal{H}(R) = (R, 0, 1, K, K_L)$. Then $\mathcal{H}(R^{a,b}) = (R, 0, ab, K, R.(b)^{-1}K_L)$.*
(5) *$Env(R^{\#}, \cdot) = Env(R^{a,b\#}, \circ_{a,b})$ and $G(R) = G(R^{a,b})$.*
(6) *If R is right distributive, then so is $R^{a,b}$.*

Proof. Part (1) follows from 1.1(1). Then (2) follows from 3.3(2) and 1.1(2), and similarly (3) follows from 1.1(3), while (4) follows from 1.1(5). Then (4) implies (5). Finally assume R is right distributive, and let K, K' be the set of right additive translations of R, $R^{a,b}$, respectively, $L = Env(R^{\#}, \cdot)$, and $L' = Env(R^{a,b\#}, \circ_{a,b})$. From (4) and (5), $K = K'$ and $L' = L$. As R is right distributive, L acts on K by 2.3 in [A1]. Thus $L' = L$ acts on $K' = K$, so R' is right distributive by another application of 2.3 in [A1]. □

3.5. *Let $R, R' \in \mathbf{R}$ and $\psi = (\alpha, \beta, \gamma) \in \mathrm{top}(R', R)$ a morphism in $\mathbf{R}^{\mathrm{top}}$. Then*

(1) *if R is right distributive, then so is R', and $\alpha : (R', +) \to (R, +)$ is an isomorphism.*
(2) *If R is left distributive, then so is R', and $\beta : (R', +) \to (R, +)$ is an isomorphism.*

Proof. Assume R is right distributive. By 3.4(3) and 3.3(1), we may assume $R' = R^{a,b}$ is a principal isotope of R, and ψ is the map in 3.4(2). Thus R' is right distributive by 3.4(6). By 1.1(4), $\alpha = R.(b)^{-1}$, so by 2.3 in [A1], α is an isomorphism.

We have established (1). Then (2) follows from (1) and the duality between R and its opposite ring, to be established in 4.8. □

4. The geometry of a ternary ring

Let $X = (R, T)$ with R a set and $T : R \times R \times R \to R$ a ternary operation on R. For example, in most instances we will take $X \in \mathbf{X} = \mathbf{R}$ or $\mathbf{T}$. Following [HP] we define a rank 2 Tits geometry $P = \mathcal{P}(X)$ as follows. Let

$$\Omega = \Omega(X) = \{\infty\} \cup R \cup (R \times R).$$

The *points* and *lines* of P are the symbols $p(\omega)$, $l(\omega)$, $\omega \in \Omega$, respectively, and the incidence relation is defined by:

$$p(\omega) \in l(\infty) \quad \text{for } \omega \in \{\infty\} \cup R,$$

$$\text{for } x \in R, \quad p(\infty) \in l(x), \qquad p(x) \in l(x, y), \quad \text{and} \quad p(x, b) \in l(x),$$

and

$$\text{for } (x, y) \in R \times R, \quad l(x, y) \text{ is incident with } p(a, b) \quad \text{iff} \quad y = T(x, a, b).$$

By construction P comes equipped with a *coordinatization* $\mathcal{C}(X)$ consisting of the functions $p, l : R \cup \{\infty\} \to P$ and $p, l : R \times R \to P$.

For $\psi = (\alpha, \beta, \gamma)$ a triple of bijections from R to R', extend $\delta \in \{\alpha, \beta, \gamma\}$ to $R \cup \{\infty\}$ by decreeing that $\delta : \infty \mapsto \infty$, and define $\partial = \mathcal{P}(\psi) : \mathcal{P}(X) \to \mathcal{P}(X')$ by

$$\partial\big(l(x)\big) = l(x\beta) \quad \text{and} \quad \partial\big(p(x)\big) = p(x\alpha) \quad \text{for } x \in R \cup \{\infty\},$$

and

$$\partial\big(l(x, y)\big) = l(x\alpha, y\gamma) \quad \text{and} \quad \partial\big(p(x, y)\big) = p(x\beta, y\gamma) \quad \text{for } (x, y) \in R \times R.$$

The ternary ring X is *planar* if $\mathcal{P}(X)$ is a projective plane, while a loop ring R is *planar* if the linear ternary ring (R, T_R) is planar. Let $\mathbf{T}^*$, $\mathbf{R}^*$ be the subcategory of planar ternary rings and planar loop rings.

4.1. *Let* $\mathbf{X} = \mathbf{R}$ *or* $\mathbf{T}$ *and* $\psi = (\alpha, \beta, \gamma) \in \mathrm{top}(X, X')$ *for some* $X, X' \in \mathbf{X}$. *Then*

(1) $\mathcal{P}(\psi) : \mathcal{P}(X) \to \mathcal{P}(X')$ *is an isomorphism of Tits geometries.*
(2) $\mathcal{P}$ *is a functor from* $\mathbf{X}^{\mathrm{top}}$ *to the category of Tits geometries and isomorphisms.*

Proof. As ψ is an isotopism, $\delta : R \to R'$ is a bijection for each $\delta \in \{\alpha, \beta, \gamma\}$ with $\delta : 0 \mapsto 0$ and $\delta : \infty \mapsto \infty$ by definition of isotopism. Thus $\mathcal{P}(\psi) = \partial : P = \mathcal{P}(R) \to P' = \mathcal{P}(R')$ is a bijection preserving type. Hence it remains to show

$$\partial\big(P\big(l(\omega)\big)\big) = P'\big(\partial\big(l(\omega)\big)\big) \quad \text{for each } \omega \in \Omega. \tag{$*$}$$

First $\partial(l(\infty)) = l(\infty)$, while $\partial(p(x)) = p(x\alpha)$, so $(*)$ holds for $\omega = \infty$. Next for $x \in R$, $\partial(l(x)) = l(x\beta)$, while $\partial(p(\infty)) = p(\infty)$ and $\partial(p(x, b)) = p(x\beta, b\gamma)$, so $(*)$ holds for $\omega \in R$. Finally let $(x, y) \in R \times R$. Then $\partial(l(x, y)) = l(x\alpha, y\gamma)$, while $\partial(p(a, b)) = p(a\beta, b\gamma)$. Further if $p(a, b) \in l(x, y)$ then $y = T(x, a, b)$, so as ψ is an isotopism, $y\gamma = T(x\alpha, a\beta, b\gamma)$, and hence $p(a\beta, b\gamma)$ is incident with $l(x\alpha, y\gamma)$. That is $\partial(P(l(x, y))) \subseteq P'(\partial(l(x, y)))$. The opposite inclusion follows by symmetry using 3.2(2). So (1) is established.

Visibly if $\bar{\psi} \in \mathrm{top}(X', \bar{X})$ then $\mathcal{P}(\psi\bar{\psi}) = \mathcal{P}(\psi)\mathcal{P}(\bar{\psi})$, so (2) holds. $\square$

Given X, X' in $\mathbf{X} = \mathbf{R}$ or $\mathbf{T}$, define $\mathrm{hom}(\mathcal{P}(X), \mathcal{P}(X'))$ to be the set of isomorphisms $\varphi : \mathcal{P}(X) \to \mathcal{P}(X')$ such that $\varphi(l(\omega)) = l(\omega)$ for $\omega \in \{\infty, 0\}$ and $\varphi(l(0, 0)) = l(0, 0)$. Set $\mathrm{aut}(\mathcal{P}(X)) = \mathrm{hom}(\mathcal{P}(X), \mathcal{P}(X))$.

4.2. *Assume* $X = (R, T)$ *and* $X' = (R', T')$ *are planar ternary rings and* $\varphi \in \mathrm{hom}(\mathcal{P}(X), \mathcal{P}(X'))$. *Then*

(1) *There exist unique bijections* α, β, *and* γ *from* R *to* R' *such that* $\varphi(p(\infty)) = p(\infty)$, $\varphi(p(a)) = p(a\alpha)$, $\varphi(p(a, 0)) = p(a\beta, 0)$, *and* $\varphi(p(0, b)) = p(0, b\gamma)$ *for all* $a, b \in R$.
(2) $\varrho(\varphi) = (\alpha, \beta, \gamma) \in \mathrm{top}(X, X')$.

Proof. As φ is an isomorphism mapping l to l', where $l = l(x) = l'$ for $x \in \{\infty, 0\}$, or $l = l(0, 0) = l'$, it follows that φ maps the points on l to the points on l'. In particular,

φ maps $l(\infty) \cap l(0) = \{p(\infty)\}$ to $l(\infty) \cap l(0) = \{p(\infty)\}$, so $\varphi(p(\infty)) = p(\infty)$. Similarly $l(\infty) \cap l(0,0) = \{p(0)\}$, so $\varphi(p(0)) = p(0)$ and $l(0) \cap l(0,0) = \{p(0,0)\}$ so $\varphi(p(0,0)) = p(0,0)$. Thus (1) follows, with $\delta : 0 \mapsto 0$ for $\delta \in \{\alpha, \beta, \gamma\}$.

Next as $P = \mathcal{P}(X)$ and $P' = \mathcal{P}(X')$ are projective planes, $l(x) = p(\infty) + p(x,0)$ for each $x \in R \cup R'$. Thus as $\varphi : P \to P'$ is an isomorphism mapping $p(\infty)$ to $p(\infty)$ and $p(x,0)$ to $p(x\beta, 0)$, $\varphi(l(x)) = l(x\beta)$. Similarly $l(0,y) = p(0) + p(0,y)$ so $\varphi(l(0,y)) = l(0, y\gamma)$. Further $p(a,b) = l(a) \cap l(0,b)$, so

$$\varphi\big(p(a,b)\big) = l(a\beta) \cap l(0, b\gamma) = p(a\beta, b\gamma).$$

Also $l(x,y) = p(x) + p(a,b)$ for each $(a,b) \in R \times R$ with $y = T(x,a,b)$. For example, $l(x,y) = p(x) + p(0,y)$, so

$$\varphi\big(l(x,y)\big) = p(x\alpha) + p(0, y\gamma) = l(x\alpha, y\gamma).$$

Then for arbitrary (a,b) with $y = T(x,a,b)$, $l(x\alpha, y\gamma) = \varphi(l(x,y))$ contains $\varphi(p(a,b)) = p(a\beta, b\gamma)$, so

$$y\gamma = T'(x\alpha, a\beta, b\gamma) \quad \text{for all } a, b \in R \text{ with } y = T(x,a,b),$$

and hence $\varrho(\varphi)$ is an isotopism, completing the proof of (2). $\square$

4.3. *Assume $X, X' \in \mathbf{T}^*$ and let $\Delta = (p(0,0), p(0), p(\infty))$. Then*

(1) *$\mathcal{P} : \mathrm{top}(X, X') \to \mathrm{hom}(\mathcal{P}(X), \mathcal{P}(X'))$ is a bijection with inverse ϱ.*
(2) *$\mathcal{P} : \mathrm{top}(X) \to \mathrm{aut}(\mathcal{P}(X))$ is an isomorphism.*
(3) *$\mathrm{aut}(\mathcal{P}(X)) = Aut(\mathcal{P}(X))_\Delta$ is the pointwise stabilizer in $Aut(\mathcal{P}(X))$ of Δ.*
(4) *$Aut(\mathcal{P}(X))_{\Delta, p(1,1)}$ is the image of $Aut(X)$ in $Aut(\mathcal{P}(X))$ under the functor $\mathcal{P}$.*
(5) *The orbit $p(1,1)Aut(\mathcal{P}(X))_\Delta$ of $p(1,1)$ under $Aut(\mathcal{P}(X))_\Delta$ is of length $|\mathrm{top}(X) : Aut(X)|$.*

Proof. Let $\psi = (\alpha, \beta, \gamma) \in \mathrm{top}(X, X')$, $P = \mathcal{P}(X)$, $P' = \mathcal{P}(X')$, and $A = Aut(P)$. By 4.1, $\partial = \mathcal{P}(\psi) : P \to P'$ is an isomorphism, and by construction ∂ maps $l(\omega)$ to $l(\omega)$ for $\omega \in \{\infty, 0\}$ and $l(0,0)$ to $l(0,0)$. Thus $\partial \in \mathrm{hom}(P, P')$, so indeed $\mathcal{P} : \mathrm{top}(X, X') \to \mathrm{hom}(P, P')$. By 4.2(2), $\varrho : \mathrm{hom}(P, P') \to \mathrm{top}(X, X')$.

Next $\partial(p(x)) = p(x\alpha)$ and $\partial(p(x,y)) = p(x\beta, y\gamma)$, so $\varrho(\partial) = (\alpha, \beta, \gamma) = \psi$; that is $\varrho \circ \mathcal{P} = 1$. Similarly let $\varphi \in \mathrm{hom}(P, P')$, and set $\psi = \varrho(\varphi) = (\alpha, \beta, \gamma)$ and $\partial = \mathcal{P}(\psi)$. Then $\partial(p(x)) = p(x\alpha) = \varphi(p(x))$ by construction of ψ in 4.2(1). Similarly $\partial(p(a,0)) = p(a\beta, 0) = \varphi(p(a,0))$ and $\partial(p(0,b)) = p(0, b\gamma) = \varphi(p(0,b))$. Further from the proof of 4.2, the isomorphisms φ and ∂ are determined by the images of these points, so $\varphi = \partial$; that is $\mathcal{P} \circ \varrho = 1$, completing the proof of (1). Then (1) and 4.1(2) imply (2).

Part (3) is a restatement of the definition of $\mathrm{aut}(P)$. By (2) and the description of $\mathcal{P}(\psi) = \partial$, $p(1,1)\partial = p(1,1)$ iff $1\beta = 1 = 1\gamma$. Suppose ∂ fixes $p(1,1)$. By 3.2(3), for all $x, y \in R$, $(xy)\gamma = x\alpha y\beta$, so

$$x\gamma = (x1)\gamma = x\alpha 1\beta = x\alpha \cdot 1 = x\alpha.$$

Thus $\alpha = \gamma$. In particular, α fixes 1, so

$$y\gamma = (1y)\gamma = 1\alpha y\beta = 1 \cdot y\beta = y\beta,$$

and hence $\beta = \gamma$. Thus $\psi \in Aut(X)$, establishing (4). Then (2)–(4) imply (5). $\square$

Remark 4.4. Define a *triangle* in a projective plane P to be an ordered triple $\Delta = (p_1, p_2, p_3)$ of noncollinear points, and write $\Gamma(\Delta)$ for the set of points not collinear with any of the lines $p_i + p_j$, $1 \leqslant i < j \leqslant 3$. A *basis* for P is an ordered 4-tuple (Δ, p_4) such that Δ is a triangle in P and $p_4 \in \Gamma(\Delta)$.

Given any set R with distinguished points $\{0, 1\}$, any basis (Δ, p_4) of P, and any bijection $f : R - \{0\} \to P(p_1 + p_3) - \{p_1, p_3\}$ with $f(1) = (p_1 + p_3) \cap (p_2 + p_4)$, we find in Chapter V of [HP] that there is a coordinatization $\mathcal{C} = \mathcal{C}(P, \Delta, p_4, R, f)$ of P in which $p_1 = p(0, 0)$, $p_2 = p(0)$, $p_3 = p(\infty)$, $p_4 = p(1, 1)$, $f(1) = p(0, 1)$, and $\Gamma(\Delta) = \{p(x, y)\colon x, y \in R\}$. Further there is a construction of a planar ternary ring $X = (R, T) = X(P, \mathcal{C})$ such that $P = \mathcal{P}(X)$ and the coordinatization $\mathcal{C}$ is $\mathcal{C}(X)$.

Let $[\mathcal{C}](P, \Delta, p_4)$ consist of the coordinatizations $\mathcal{C}(P, \Delta, R, p_4, f)$. We see in part (1) of the next lemma that if $\mathcal{C}_i \in [\mathcal{C}](P, \Delta, p_4)$ for $i = 1, 2$, then $X(P, \mathcal{C}_1)$ is isomorphic to $X(P, \mathcal{C}_2)$; we write $[X](P, \Delta, p_4)$ for the isomorphism class of ternary rings $X(P, \mathcal{C})$ with $\mathcal{C} \in [\mathcal{C}](P, \Delta, p_4)$, and set

$$[X](P, \Delta) = \bigcup_{p \in \Gamma(\Delta)} [X](P, \Delta, p).$$

We see in part (3) of the next lemma that $[X](P, \Delta)$ is an isotopy class.

4.5. *Let Δ, $\bar{\Delta}$ be triangles in projective planes P, $\bar{P}$, respectively, and let $p \in \Gamma(\Delta)$.*

(1) *Suppose $\psi : P \to P$ is an isomorphism with $\Delta\psi = \bar{\Delta}$. Let $\bar{p} = p\psi$ and $\mathcal{C} = \mathcal{C}(P, \Delta, p, R, f)$ and $\bar{\mathcal{C}} = \mathcal{C}(\bar{P}, \bar{\Delta}, \bar{p}, \bar{R}, \bar{f})$ be coordinatizations. Define $\varphi : R \to \bar{R}$ by $0\varphi = 0$ and $\varphi = f\psi\bar{f}^{-1}$ as a map from $R^{\#}$ to $\bar{R}^{\#}$. Then $\varphi : X(P, \mathcal{C}) \to X(\bar{P}, \mathcal{C})$ is an isomorphism such that for all $x, y \in R$, $l(x)\psi = l(x\varphi)$, $p(x)\psi = p(x\varphi)$, $l(x, y)\psi = l(x\varphi, y\varphi)$, and $p(x, y)\psi = p(x\varphi, y\varphi)$.*

(2) *The following are equivalent*:
   (i) *There exists an isomorphism $\psi : P \to \bar{P}$ with $\Delta\psi = \bar{\Delta}$.*
   (ii) *$[X](P, \Delta, p) = [X](\bar{P}, \bar{\Delta}, \bar{p})$ for some $\bar{p} \in \Gamma(\bar{\Delta})$.*
   (iii) *$[X](P, \Delta) = [X](\bar{P}, \bar{\Delta})$.*

(3) *$[X](P, \Delta)$ is the set of isomorphism classes in an isotopy class of ternary rings.*

Proof. Parts (1) and (2) are essentially Exercises 5.2 and 5.3 in [HP], but we supply a few details.

Let $x, y, a, b \in R$. Check that by construction of $\mathcal{C}$, $\bar{\mathcal{C}}$, and φ, $p(a, b)\psi = p(a\varphi, b\varphi)$, $l(x)\psi = l(x\varphi)$, etc. Then $xa + b = y$ iff $p(a, b) \in l(x, y)$ iff $p(a\varphi, b\varphi) = p(a, b)\psi \in l(x, y)\psi = l(x\varphi, y\varphi)$ iff $x\varphi \cdot a\varphi + b\varphi = y\varphi$, so φ is an isomorphism. That is (1) holds.

By (1), (2)(i) implies (2)(ii). Assume (2)(ii) holds. Then there exists $X \in [X](P, \Delta, p)$ and $\bar{X} \in [X](\bar{P}, \bar{\Delta}, \bar{p})$ with $P = \mathcal{P}(X)$, $\bar{P} = \mathcal{P}(\bar{X})$, and $\Delta, \bar{\Delta}$ equal to $(p(0, 0), p(0), p(\infty))$ in $P, \bar{P}$, respectively; and there is an isomorphism $\varphi : X \to \bar{X}$. Now $\varphi \in \mathrm{top}(X, \bar{X})$, so by 4.1, $\psi = \mathcal{P}(\varphi) : P \to \bar{P}$ is an isomorphism, and by definition of the functor $\mathcal{P}$, $\Delta\psi = \bar{\Delta}$. Thus (2)(ii) implies (2)(i). Finally letting p vary over $\Gamma(\Delta)$, (2)(ii) is equivalent to (2)(iii).

Let $X = X(P, \mathcal{C})$; thus $P = \mathcal{P}(X)$ and $\Delta = (p(0,0), p(0), p(\infty))$. Let $\bar{X}$ be a ternary ring and $\bar{P} = \mathcal{P}(\bar{X})$. If $\phi \in \mathrm{top}(X, \bar{X})$ then by 4.3(1), $\partial = \mathcal{P}(\phi)$ is an isomorphism from P to $\bar{P}$ with $\Delta\partial = (p(0,0), p(0), p(\infty))$ in $\bar{P}$. Hence $\bar{X} \in X[\bar{P}, \Delta] = X[P, \Delta]$ by (2). Conversely if $\bar{X} \in [X](P, \Delta)$ then $\bar{X} = X(P, \bar{\mathcal{C}})$ for some coordinatization $\bar{\mathcal{C}}$ at Δ, so the identity map $1_P \in \mathrm{hom}(\mathcal{P}(X), \mathcal{P}(\bar{X}))$, and X and $\bar{X}$ are isotopic by 4.3(1). This completes the proof of (3). □

4.6. *Let P be a projective plane, Δ a triangle in P, $\Gamma = \Gamma(\Delta)$, $A = Aut(P)$, and $B = A_\Delta$ the pointwise stabilizer of Δ in A. Then*

(1) $B \cong \mathrm{top}(X)$ *for* $X \in [X](P, \Delta)$.
(2) *$p, q \in \Gamma$ are conjugate under B iff* $[X](P, \Delta, p) = [X](P, \Delta, q)$.
(3) *B has $|[X](P, \Delta)|$ orbits on Γ.*
(4) *For $p \in \Gamma$, $B_p \cong Aut(X)$ and $|pB| = |\mathrm{top}(X) : Aut(X)|$ for $X \in [X](P, \Delta, p)$.*
(5) *A triangle $\bar{\Delta}$ in P is conjugate to Δ under A iff* $[X](P, \Delta) = [X](P, \bar{\Delta})$.

Proof. Part (1) follows from parts (2) and (3) of 4.3. Parts (2) and (5) follow from 4.5(2), while (2) implies (3) and parts (4) and (5) of 4.3 imply (4). □

4.7. *Let P be a projective plane, $\Delta = (p_1, p_2, p_3)$ a triangle in P, $I = \{p_1, p_2, p_3\}$, $S = Sym(I)$ the symmetric group on I, and Σ the set of triangles of P with entries from I. Represent S on Σ via $(x, y, z)s = (xs, ys, zs)$ for $s \in S$, and define a relation $\sim$ on S by $s \sim t$ if $[X](P, \Delta s) = [X](P, \Delta t)$. Then*

(1) *The map $s \mapsto \Delta s$ is an equivalence of the representation of S on itself by right multiplication with its representation on Σ.*
(2) *For $t \in S$, the following are equivalent*:
  (i) *The action of t on I extends to an automorphism of P.*
  (ii) $1 \sim t$.
  (iii) *$s \sim st$ for all $s \in S$.*
(3) *$\sim$ is congruence* mod S^+, *where S^+ is the equivalence class of* 1 *under $\sim$ and a subgroup of S. Thus S acts transitively on $S/{\sim}$ and the map $sS^+ \mapsto \tilde{s}$ is an equivalence of the representation of S by right multiplication on S/S^+ with it representation on $\tilde{S}$ via $t : \tilde{s} \mapsto \widetilde{ts}$.*
(4) $S^+ = Aut_{Aut(P)}(\Delta)$.
(5) *There are* $|S : S^+| = 1$, 2, 3, *or* 6 *isotopy classes of ternary rings in* $\{[X](P, \bar{\Delta}) \colon \bar{\Delta} \in \Sigma\}$.

Proof. Part (1) is trivial, and (2) follows from 4.6(5). Then (2) implies (3), while (3) implies (4) and (5). □

Given $X \in \mathbf{X} = \mathbf{R}$ or $\mathbf{T}$, let $P^* = \mathcal{P}^*(X)$ be the dual of P; that is P^* is the Tits geometry P, except with the role of points and lines reversed. We write $p^*(x)$, $p^*(x, y)$ for the lines $l(x)$, $l(x, y)$ of P regarded as points of P^*, and write $l^*(x)$, $l^*(x, y)$ for the points $p(x)$, $p(x, y)$ of P regarded as lines of P^*.

Given $R \in \mathbf{R}$, let $R^{\mathrm{op}} = (R, \boxplus, *)$ denote the *opposite ring* to R, which has multiplication defined by $x * y = y \cdot x$ for $x, y \in R$, and addition defined by $L_\boxplus(z) = L_+(z)^{-1}$ for $z \in R$.

4.8. *Let $R \in \mathbf{R}$. Then*

(1) $R^{\mathrm{op}} \in \mathbf{R}$ *with zero* 0 *and multiplicative identity* 1.
(2) (i) $(R^{\mathrm{op}\#}, *)$ *is the opposite loop to* $(R^{\#}, \cdot)$*, so* $(R^{\#}, \cdot)$ *is a group iff* $(R^{\mathrm{op}\#}, *)$ *is a group.*
 (ii) $(R, +)$ *is a group iff* $(R^{\mathrm{op}}, \boxplus)$ *is a group, in which case* $+ = \boxplus$.
 (iii) R *is right distributive iff* R^{op} *is left distributive.*
 (iv) R *is planar iff* R^{op} *is planar.*
(3) *Incidence in* $\mathcal{P}^*(R)$ *is given by*:
 (i) $l^*(\infty)$ *is incident with the points* $p^*(\omega)$, $\omega \in \Omega$.
 (ii) *For* $a \in R$, $l^*(a)$ *is incident with the points* $p^*(\infty)$ *and* $p^*(a, y)$, $y \in R$.
 (iii) *For* $a, b \in R$, $l^*(a, b)$ *is incident with the points* $p^*(a)$ *and* $p^*(x, y)$*, such that* $y = a * x + b$.
(4) *The map* $p(x) \mapsto p^*(x)$, $l(x) \mapsto l^*(x)$ *for* $x \in R \cup \{\infty\}$ *and* $p(a, b) \mapsto p^*(a, b)$, $l(a, b) \mapsto l^*(a, b)$ *for* $(a, b) \in R \times R$ *is an isomorphism of* $\mathcal{P}(R^{\mathrm{op}})$ *with* $\mathcal{P}^*(R)$.
(5) $(R^{\mathrm{op}})^{\mathrm{op}} = R$.
(6) *If* $\psi = (\alpha, \beta, \gamma) \in \mathrm{top}(R, \bar{R})$ *then* $\psi^{\mathrm{op}} = (\beta, \alpha, \gamma) \in \mathrm{top}(R^{\mathrm{op}}, \bar{R}^{\mathrm{op}})$.

Proof. Part (5) is trivial, and (3) is immediate from the definition of $\mathcal{P}^*$ and multiplication in R^{op}.

Part (i) of (2) follows from the definition of the binary operation $*$. As $(R, +)$ is a loop, for each $z \in R$, $L_+(z)$ is a permutation, so $L_{\boxplus}(z) = L_+(z)^{-1}$ is a permutation. Further $L_+(0) = 1$, so $L_{\boxplus}(0) = L_+(0)^{-1} = 1$, and hence 0 is an identity for $(R^{\mathrm{op}}, \boxplus)$. Next as $(R, +)$ is a loop, given x, y there exists a unique z with $yL_+(z) = x$. Hence there exists a unique z with $y = xL_+(z)^{-1}(z) = z \boxplus x$, so $R_{\boxplus}(x)$ is a permutation. Therefore $(R^{\mathrm{op}}, \boxplus)$ is a loop, completing the proof of (1).

Observe R is right distributive iff for all $x \in R$ and $y \in R^{\#}$, $L_+(x)^{R.(y)} = L_+(xy)$ iff $L(R) = \langle R.(y)\colon y \in R^{\#}\rangle$ acts on $\hat{K}(R) = \{L_+(x)\colon x \in R\}$ iff $L(R)$ acts on $\hat{K}(R)^{-1} = \{L_+(x)^{-1}\colon x \in R\}$. Similarly R is left distributive iff $\hat{L}(R) = \langle L.(y)\colon y \in R^{\#}\rangle$ acts on $\hat{K}(R)$. But $\hat{K}(R)^{-1} = \hat{K}(R^{\mathrm{op}})$ and $\hat{L}(R^{\mathrm{op}}) = L(R)$, so (2)(iii) follows.

If $(R, +)$ is a group then $(R, +)$ is an abelian group. Then for $x, y \in R$, $L_{\boxplus}(x \boxplus y)$ is the unique member of $\hat{K}(R^{\mathrm{op}})$ such that $0L_{\boxplus}(x \boxplus y) = 0L_{\boxplus}(x)L_{\boxplus}(y)$. But

$$0L_{\boxplus}(x)L_{\boxplus}(y) = 0L_+(x)^{-1}L_+(y)^{-1} = (-x) + (-y) = 0L_+(x + y)^{-1} = 0L_{\boxplus}(x + y),$$

so indeed $x \boxplus y = x + y$. Thus (2)(ii) holds.

To check that (4) holds amounts to checking that $p(x, y)$ is incident with $l(a, b)$ in $\mathcal{P}(R^{\mathrm{op}})$ iff $p^*(x, y)$ is incident with $l^*(a, b)$ in $\mathcal{P}^*(R)$. From (4), this is equivalent to $y = a * x + b$ iff $b = a * x \boxplus y$. It suffices to show that for all y, z, b in R, $y = z + b$ iff $b = z \boxplus y$. But $y = z + b$ iff $y = bL_+(z)$ iff $b = yL_+(z)^{-1}$, while $b = z \boxplus y$ iff $b = yL_{\boxplus}(z)$, (4) follows as $L_{\boxplus}(z) = L_+(z)^{-1}$ by definition of $\boxplus$.

Note that $\mathcal{P}(R)$ is a plane iff $\mathcal{P}^*(R)$ is a plane, so (4) implies (2)(iv).

Assume the hypothesis of (6). Claim for $x, y \in R$,

$$L_+(x * y)^{-1}\gamma = \gamma L_+(x\beta * y\alpha)^{-1}.$$

Namely for $z \in R$,

$$z\gamma L_+(x\beta * y\alpha) = (x\beta * y\alpha) + z\gamma = (y\alpha \cdot x\beta) + z\gamma = (y \cdot x + z)\gamma$$
$$= (x * y + z)\gamma = zL_+(x * y)\gamma,$$

so $\gamma L_+(x\beta * y\alpha) = L_+(x * y)\gamma$, establishing the claim.

Now by the claim:

$$(x * y \boxplus z)\gamma = zL_{\boxplus}(x * y)\gamma = zL_+(x * y)^{-1}\gamma$$
$$= z\gamma L_+(x\beta * y\alpha)^{-1} = z\gamma L_{\boxplus}(x\beta * y\alpha) = x\beta * y\alpha \boxplus z\gamma,$$

proving (6). □

4.9. *Let $X = (R, T) \in \mathbf{T}^*$, and let $P = \mathcal{P}(X)$, $\Delta = \{p(0,0), p(0), p(\infty)\} \subseteq P$ and $p_4 = p(1,1)$. Assume P' is a subplane of P containing $\Delta \cup \{p_4\}$, and set $R' = \{x \in R\colon\ p(0,x) \in P'\}$ and let $T' = T|_{R'}\colon\ R' \times R' \times R' \to R$. Then*

(1) *$X' = (R', T') \in \mathbf{T}^*$ with $\mathcal{P}(X') = P'$.*
(2) *R' is a loop subring of R.*
(3) *If T is linear then so is T'.*
(4) *If P' is Desarguesian then R' is a field and X' is linear.*
(5) *Assume $\bar{X} = (\bar{R}, \bar{T})$ is a ternary subring of X and let*

$$\bar{P} = \bigl\{p(\infty), l(\infty), p(x), l(x), p(x,y), l(x,y)\colon\ x, y \in \bar{R}\bigr\}.$$

Then $\bar{P}$ is a subplane of P isomorphic to $\mathcal{P}(\bar{X})$. If X is linear, so is $\bar{X}$, and if $\bar{X}$ is linear and $\bar{R}$ is a field then $\bar{P}$ is Desarguesian.

Proof. Let $\mathcal{C} = \mathcal{C}(P, \Delta, p_4, R, f)$ be the coordinatization of P by X, and let f' be the restriction of f to R'. By definition of $\mathcal{C}$ in Chapter V of [HP]:

(a) The points and lines of P' are of the form $p(\infty)$, $l(\infty)$, $p(x)$, $l(x)$, $p(x,y)$, $l(x,y)$, for $x, y \in R$.

Then by (a), $T'(R') \subseteq R'$, and appealing again to the definition of coordinatization in [HP]:

(b) $\mathcal{C}' = \mathcal{C}(P', \Delta, p_4, R', f')$ coordinatizes P' with $X' = X(P', \mathcal{C}')$.

In particular, (1) follows from (b).

Next by definition of $+$ and $\cdot$ in the loop ring R of the ternary ring $X = (R, T)$, for $a, b \in R$, $ab = T(a,b,0)$ and $a + b = T(1,a,b)$. Similarly the addition $+'$ and multiplication $\cdot'$ in R' are defined by

$$a \cdot' b = T'(a,b,0) = T(a,b,0) = ab \quad \text{and} \quad a +' b = T'(1,a,b) = T(1,a,b) = a + b,$$

establishing (2). As $T = T|_{R'}$, (3) holds. Finally if P' is Desarguesian then all coordinatizations of P' are linear coordinatizations by fields, so (4) holds.

Part (5) follows using similar arguments. □

5. Elations and quasifields

Recall $\mathbf{T}^*$ is the subcategory of $\mathbf{T}$ consisting of the planar ternary rings, and $\mathbf{R}^*$ is the subcategory of $\mathbf{R}$ consisting of the planar loop rings.

With the exception of the last lemma in the section, the results in this section are well known, but sometimes there is not a good reference for a result, or the existing proof is overly complicated.

5.1. *$R \in \mathbf{R}$ is a right quasifield iff R^{op} is a left quasifield.*

Proof. This follows from 4.6(2). □

Remark 5.2. We recall some facts from Chapter IV of [HP] and Section 20.4 of [H]: Let (p, l) be an incident point–line pair in a projective plane P. An *elation* with *center* p and *axis* l is an automorphism γ of P which fixes each point on l and each line through p. It turns out (cf. Theorem 20.4.1 in [H]) that if γ is a nonidentity elation then γ fixes no further points or lines and the set $E(p, l)$ of such elations is a subgroup of $Aut(\mathcal{P})$. Then $E(p, l)$ is semiregular on $m - \{p\}$ and $q - \{l\}$ for each line $m \neq l$ through p and each point $q \neq p$ on l, so if P is finite then $|E(p, l)|$ divides $|R|$.

The plane P is said to be (p, l)-*transitive* if $E(p, l)$ is transitive on each of the sets $m - \{p\}$ and $q - \{l\}$ of the previous paragraph; if P is finite this is equivalent to $|E(p, l)| = |R|$. Further P is a *translation plane* with axis l if P is (p, l)-transitive for all points p on l, and P is a *dual translation plane* with center p if P is (p, l)-transitive for each line l through p.

Let $\mathcal{Q} = \mathcal{Q}(P)$ be the set of points q of P such that P is a dual translation plane with center q.

5.3.

(1) *If $X = (R, T) \in \mathbf{T}^*$ then $\mathcal{P}(X)$ is $(p(\infty), l(\infty))$-transitive iff $T = T_R$ is linear (so that $\mathcal{P}(X) = \mathcal{P}(R)$) and $(R, +)$ is a group.*
(2) *A loop ring R is a left quasifield iff $\mathcal{P}(R)$ is a translation plane with axis $l(\infty)$.*
(3) *A loop ring R is a right quasifield iff $\mathcal{P}(R)$ is a dual translation plane with center $p(\infty)$.*

Proof. Part (1) is Theorem 6.2 in [HP] or Theorem 2.4.5 in [H]. Part (2) is Theorem 6.3 in [HP]. See also Theorem 20.4.6 in [H], although there the difference between the coordinatization of planes between the two texts causes the result in [H] to be dual to that of [HP]. Finally (3) follows from (2) and 5.1. □

5.4. *Assume R is a finite right quasifield, let $P = \mathcal{P}(R)$, $A = Aut(P)$, and $E = E(p(\infty))$ the group of elations of P with center $p(\infty)$. Then*

(1) *$|R| = p^e$ for some prime p, and E is an abelian p-group of order p^{2e} and exponent p.*
(2) *$p(\infty)$ is the unique point of P fixed by E.*
(3) *The set $P(p(\infty))$ of $p^e + 1$ lines through $p(\infty)$ is the set of fixed lines of E, and E is regular on the set Θ of lines not incident with $p(\infty)$.*
(4) *E has $p^e + 1$ orbits on points of P distinct from $p(\infty)$: The sets $P(l) - \{p(\infty)\}$, $l \in P(p(\infty))$, each of order p^e.*

(5) *The stabilizer* $B = A_{p(\infty),m}$ *in* A *of* $p(\infty)$ *and a line* $m \in \Theta$ *is a complement to* E *in* $A_{p(\infty)}$, *and the representation of* B *on* E *via conjugation is equivalent to its representation on* Θ.
(6) *Let* (q,l) *be an incident point–line pair,* $\Delta^{q,l} = \{p_1, p_2, p_3\}$ *a triangle with* $p_3 = q$ *and* $p_2 + p_3 = l$, $s \in \Gamma(\Delta^{q,l})$, *and* $X^{q,l} = (R^{q,l}, T^{q,l}) \in [X](P, \Delta^{q,l}, s)$. *Then* $P = \mathcal{P}(X^{q,l})$ *and* $q \in \mathcal{Q}$ *iff* $T^{q,l}$ *is linear and* $R^{q,l}$ *is a right quasifield.*

Proof. By Theorem 20.4.3 in [H], E is abelian of exponent p. By hypothesis $|E(p^\infty, l)| = p^e$ for each l through $p(\infty)$. Recall from Remark 5.2 that $E(p^\infty, l)$ is regular on $m - \{p(\infty)\}$ for $m \neq l$ through $p(\infty)$, so $|E(p^\infty, l)E(p^\infty, m)| = p^{2e}$. Further the set Θ of lines not incident with $p(\infty)$ is of order p^{2e} and E is semiregular on Θ, so we conclude E is regular on Θ and $E = E(p(\infty), l)E(p(\infty), m)$ is of order p^{2e}. Hence (1) and (3) hold. Then (1) and (3) implies (2) and (4), and (3) implies (5).

Pick (q,l) as in (6). By definition of $\mathcal{Q}$, $q \in \mathcal{Q}$ iff P is a dual translation plane with center q. From Remark 4.4, $P = \mathcal{P}(X^{q,l})$; then (6) follows from 5.3(3). □

5.5. *Assume* R *is a right quasifield and let* $P = \mathcal{P}(R)$ *and* $A = Aut(P)$. *Then the following are equivalent*:

(1) *A does not fix* $p(\infty)$.
(2) $\mathcal{Q} \neq \{p(\infty)\}$.
(3) *There exists a right quasifield* R' *and an isomorphism* $\varphi : \mathcal{P}(R') \to P$ *such that* $\varphi(p(\infty)') \neq p(\infty)$.
(4) R *is a field and* P *is the plane of a* 3*-dimensional vector space* V *over* R.

Proof. We first observe that A is transitive on $\mathcal{Q}$: Let $q \in \mathcal{Q}$ and S and Q Sylow p-subgroups of A containing E and $E(q)$, respectively. By Sylow's theorem there is $g \in A$ with $S^g = Q$. By 5.4(2), $p(\infty)$ and q are the unique fixed points of S and Q, respectively, so $p(\infty)g = q$, establishing the observation.

Notice the observation shows (1) and (2) are equivalent. Further 5.4(6) shows (2) and (3) are equivalent. Moreover, if (4) holds then A is the group $P\Gamma(V)$ of projective semilinear maps on V, so $\mathcal{Q}$ is the set of all points of P and hence (2) holds. Thus we may assume (2) holds and it remains to show (4) holds.

Let $q_1 = p(\infty)$ and $q_2 = q \in \mathcal{Q} - \{q_1\}$. Let $l = p(\infty) + q$ and $L = \langle E(q_1), E(q_2)\rangle$.

Next L fixes l, and from the proof of 5.4, $E = E(l) = E(q_1, l)E(q_2, l)$ is the full group of elations on l, so P is a translation plane with axis l by 5.3(2). Let r be a point not on l. From the dual of 5.4(5), the stabilizer $B = A_{l,r}$ of (l, r) in A is a complement to E in A_l. Let $U = B \cap L$, $m_i = q_i + r$, and $E_i = E(q_i, m_i)$ for $i = 1, 2$. Observe $U = \langle E_1, E_2\rangle$ and E_i is regular on $P(l) - \{q_i\}$ and normal in U_{q_i}, with $E_i \cong E_{p^e}$. Thus U is 2-transitive on $P(l)$, so from the classification of the finite doubly transitive groups, U induces $L_2(p^e)$ on $P(l)$.

By the dual of 5.4(3), E is regular on the set of points not on l, so $C_U(E)$ fixes all such points and hence is trivial. Thus U is faithful on E. Further $C_E(E_i) = E(q_i, l) \cong E_{p^e}$, so we conclude $U \cong SL_2(p^e)$ and E is the natural module for U. Thus, in the language of Section VII.3 of [HP], the congruence partition $\mathcal{S} = \{E(l, q)\colon q \in l\}$ of E is the set of 1-dimensional $\mathbf{F}_{p^e}$-subspaces of E viewed as a 2-dimensional $\mathbf{F}_{p^e}$-space for U, so (4) follows from Theorem 7.7 in [HP]. □

5.6. *Assume* R *is a finite right quasifield and* $(\bar{R}, \bar{T})$ *is a ternary ring isotopic to* (R, T_R). *Then*

(1) $\bar{T} = T_{\bar{R}}$ *is linear and* $\bar{R}$ *is isotopic to* R.
(2) $\bar{R}$ *is a right quasifield.*

(3) *$\bar{R}$ is isomorphic to a principal isotope $R^{u,v}$ of R.*
(4) *Let $\mathcal{H}(R) = (R, 0, 1, K, K_L)$. Then $\mathcal{H}(R^{u,v}) = (R, 0, uv, K, R.(v)^{-1}K_L)$.*

Proof. Part (1) follows from 3.2(4). As R is a right quasifield, R is right distributive and $(R, +)$ is a group. Hence $\bar{R}$ also satisfied these properties by 3.5(1), so (2) holds. Parts (3) and (4) follow from the corresponding parts of 3.4. □

6. Reflections

In this section assume $R = (R, +, \cdot)$ is a right distributive planar loop ring, and let $P = \mathcal{P}(R)$ with basic triangle $\Delta = (p(0,0), p(0), p(\infty))$. Let $\Delta' = (p(0), p(0,0), p(\infty))$. Set $\mathcal{H}(R) = (R, 0, 1, K, K_L)$.

A *reflection* of R is a ternary ring in the isotopism class $[X](P, \Delta')$. Notice that, in the notation of 4.7, $\Delta' = \Delta t$, where $t = (p(0), p(0,0))$ is the transposition in the symmetric group on $\{p(0,0), p(0), p(\infty)\}$, which reflects the triangle Δ about the axis through $p(\infty)$ perpendicular to the line $l(0,0)$ opposite to $p(\infty)$ in the triangle Δ. Observe also that by 4.3(1):

6.1. *Let $\bar{X} = (\bar{R}, \bar{T})$ be a ternary ring and $\bar{P} = \mathcal{P}(\bar{X})$ with basic triangle $\bar{\Delta}$. Then all reflections of R are isotopic and $\bar{R}$ is a reflection of R iff there exists an isomorphism $\varphi : \bar{P} \to P$ with $\bar{\Delta}\varphi = \Delta'$.*

6.2. *Assume $\oplus$ and $*$ are binary operations on R such that 0 is an identity for $(R, \oplus)$ and for all $x \in R$, $0 * x = x * 0 = 0$. Let $\bar{R} = (R, \oplus, *)$ and $\bar{P} = \mathcal{P}(\bar{R})$. For $x \in R$, write $-x$ for the unique element of R with $x + (-x) = 0$, and write I for the permutation $x \mapsto -x$ of R. Assume $\iota \in Sym(R^\#)$ with $\iota^{-1}(1)$ an identity for $(R^\#, *)$, and extend ι to R by $\iota(0) = 0$. Define $\chi = \chi_\iota : \bar{P} \to P$ by: χ maps $\bar{p}(\infty)$ to $p(\infty)$, $\bar{l}(\infty)$ to $l(0)$, $\bar{l}(0)$ to $l(\infty)$, and for $x, y \in R$ and $w \in R^\#$, χ maps $\bar{l}(w)$ to $l(\iota(w))$, $\bar{p}(w, y)$ to $p(\iota(w), y\iota(w))$, $\bar{p}(x)$ to $p(0, -x)$, $\bar{p}(0, x)$ to $p(-x)$, and $\bar{l}(x, y)$ to $l(-y, -x)$. Then*

(1) *$IR.(u)I^{-1} = R.(u)$ for each $u \in R^\#$.*
(2) *χ is an isomorphism iff* (i) *and* (ii) *hold*:
   (i) *For each $a \in R^\#$, $R_*(a) = R.(\iota(a))^{-1}$.*
   (ii) *For each $b \in R$, $R_\oplus(b) = IR_+(b)^{-1}I^{-1}$.*
(3) *If χ is an isomorphism then $\bar{R}$ is a right distributive planar loop ring with $\mathcal{H}(\bar{R}) = (R, 0, \iota^{-1}(1), K^{-I^{-1}}, K_L^{-1})$.*

Proof. Let $x \in R$ and $u \in R^\#$. Then $xIR.(u) = (-x)u$ and $xR.(u)I = -(xu)$. Further as R is right distributive,

$$xu = (-x)u = \bigl(x + (-x)\bigr)u = 0 \cdot u = 0,$$

so $(-x)u = -(xu)$, establishing (1).

Visibly $\chi : \bar{P} \to P$ is a bijection, so it remains to check:

$$\chi\bigl(\bar{P}(l)\bigr) = P\bigl(\chi(l)\bigr) \quad \text{for each line } l \text{ of } \bar{P}. \tag{$*$}$$

First $\bar{P}(\bar{l}(\infty)) = \{\bar{p}(x)\colon x \in R \cup \{\infty\}\}$ and $\chi(\bar{l}(\infty)) = l(0)$ with

$$P\big(l(0)\big) = \big\{p(\infty), p(0,x)\colon x \in R\big\}.$$

So as χ maps $\bar{p}(\infty)$ to $p(\infty)$, and $\bar{p}(x)$ to $p(0,-x)$ for $x \in R$, $(*)$ holds for $l = \bar{l}(\infty)$. Similarly $(*)$ holds when l is $\bar{l}(0)$.

Next for $w \in R^\#$, $\bar{P}(l(w)) = \{\bar{p}(\infty), \bar{p}(w,y)\colon\ y \in R\}$ and $\chi(\bar{l}(w)) = l(\iota(w))$. So as $\chi(\bar{p}(\infty)) = p(\infty)$, and as $\chi(\bar{p}(w,y)) = p(\iota(w), y\iota(w))$, $(*)$ holds for $\bar{l}(w)$.

Finally let $x, y \in R$. Then $\bar{P}(\bar{l}(x,y)) = \{\bar{p}(x)\} \cup \{\bar{p}(a,b)\colon\ y = x * a \oplus b\}$ contains $\bar{p}(0,y)$. Also $\chi(\bar{l}(x,y)) = l(-y,-x)$, while $\chi(\bar{p}(x)) = p(0,-x)$, and $\chi(\bar{p}(0,y)) = p(-y)$ are incident with $l(-y,-x)$. Thus as $\chi(\bar{p}(a,b)) = p(\iota(a), b\iota(a))$, it remains to show that if $a \in R^\#$ and $b \in R$ with $y = x * a \oplus b$ then also $-x = (-y)\iota(a) + b\iota(a)$.

As R is right distributive,

$$(-y)\iota(a) + b\iota(a) = \big((-y) + b\big)\iota(a) = yIR_+(b)R_\cdot\big(\iota(a)\big),$$

so $-x = (-y)\iota(a) + b\iota(a)$ iff $x = yIR_+(b)R_\cdot(\iota(a))I^{-1}$. On the other hand, $y = x * a \oplus b = xR_*(a)R_\oplus(b)$, which holds iff $x = yR_\oplus(b)^{-1}R_*(a)^{-1}$. Thus to prove (2), it remains to show that χ is an isomorphism iff

$$\text{for all } a \in R^\# \text{ and } b \in R, \quad R_\oplus(b)^{-1}R_*(a)^{-1} = IR_+(b)R_\cdot\big(\iota(a)\big)I^{-1}. \tag{!}$$

Specialize (!) to $b = 0$. As 0 is an additive identity for R and $\bar{R}$, (!) becomes $R_*(a)^{-1} = IR_\cdot(\iota(a))I^{-1} = R_\cdot(\iota(a))$ by (1). Thus (!) holds when $b = 0$ iff condition (i) of (2) holds.

Next specialize (!) to $\iota(a) = 1$. As 1 and $\iota^{-1}(a)$ are the multiplicative identities of R and $\bar{R}$, respectively, (!) becomes $R_\oplus(b)^{-1} + IR_+(b)I^{-1}$, which is condition (ii) of (2). Therefore if χ is an isomorphism then conditions (i) and (ii) hold. Conversely if these conditions hold, then by (1):

$$\begin{aligned} IR_+(b)R_\cdot\big(\iota(a)\big)I^{-1} &= IR_+(b)I^{-1} \cdot IR_\cdot\big(\iota(a)\big)I^{-1} \\ &= IR_+(b)I^{-1} \cdot R_\cdot\big(\iota(a)\big) = R_\oplus(b)^{-1}R_*(a)^{-1}, \end{aligned}$$

so that (!) holds and hence χ is an isomorphism. This completes the proof of (2).

Finally assume χ is an isomorphism. Then $\bar{P} \cong P$, so $\bar{P}$ is a plane, and hence $\bar{R}$ is a planar loop ring. By (2), $\mathcal{H}(\bar{R}) = (R, 0, \iota^{-1}(1), K^{-I^{-1}}, K_L^{-1})$. By (1), $K_L^{-I^{-1}} = K_L^{-1}$, so (3) holds. $\square$

6.3.

(1) *The reflections of R are of the form $(R', T_{R'})$, where R' is a planar right distributive loop ring, and R' is isotopic to $\bar{R} = (R, \oplus, *)$, where in the notation of* 6.2*, for each $a \in R^\#$, $R_*(a) = R_\cdot(a)^{-1}$, and for each $b \in R$, $R_\oplus(b) = IR_+(b)^{-1}I^{-1}$.*

(2) *$G(\bar{R}) = G(R)^{I^{-1}}$, $Env(\bar{R}^\#, *) = Env(R^\#, \cdot)$, and $\mathcal{H}(\bar{R}) = (R, 0, 1, K^{-I^{-1}}, K_L^{-1})$.*

(3) *If R is a right quasifield then $\oplus = +$, $K^{-I^{-1}} = K$, and $G(R) = G(\bar{R})$.*

(4) *Assume R is a right nearfield and let ι be the inversion map $\iota(a) = a^{-1}$ on R. Then $\chi = \chi_\iota$ is an involutory automorphism of P acting as $(p(0), p(0,0))$ on $\{p(0,0), p(0), p(\infty)\}$.*

Proof. Define $\bar{R}$ as in (1), let ι be the identity map on R, and define $\chi = \chi_\iota$ as in 6.2. By 6.2(2), χ is an isomorphism of $\bar{P} = \mathcal{P}(\bar{R})$ with P such that $\bar{\Delta} = \Delta'$, so by 6.1, $\bar{X} = (\bar{R}, T_{\bar{R}})$ is a reflection of R, and all reflections of R are isotopic. Further the last statement in (2) follows from 6.2(3). From that statement,

$$Env\big(\bar{R}^\#, *\big) = \big\langle K_L^{-1}\big\rangle = \langle K_L\rangle = Env\big(R^\#, \cdot\big),$$

and then using 6.2(1),

$$G(\bar{R}) = \big\langle K^{-I^{-1}}, Env\big(\bar{R}^\#, *\big)\big\rangle = \big\langle K, Env\big(R^\#, \cdot\big)\big\rangle^{I^{-1}} = G(R)^{I^{-1}},$$

completing the proof of (2).

Assume R is a right quasifield. Then for $b, x \in R$, $R_+(b)^{-1} = R_+(-b)$ and

$$xIR_+(b)^{-1}I^{-1} = -\big((-x)R_+(-b)\big) = -(-x-b) = x+b = xR_+(b),$$

so $\oplus = +$, and $K^{-I^{-1}} = K$, so $G(R) = G(\bar{R})$ from the discussion in the previous paragraph. This establishes (3).

Assume R is a right nearfield, and take $\bar{R} = R$ and ι the inversion map $\iota(a) = a^{-1}$. Then $(R^\#, \cdot)$ is a group, so $R.: R^\# \to Sym(R^\#)$ is an injective group homomorphism. Therefore $R.(\iota(a))^{-1} = R.(a^{-1})^{-1} = R.(a)$, so 6.2(2) implies (4). □

7. Shifts

In this section assume $R = (R, +, \cdot)$ is a right quasifield, and let $P = \mathcal{P}(R)$ with basic triangle $\Delta = (p(0,0), p(0), p(\infty))$. For $w \in R^\#$, let $\Delta^w = (p(0,w), p(0), p(\infty))$. Set $\mathcal{H}(R) = (R, 0, 1, K, K_L)$.

A *shift* of R to $w \in R^\#$ is a ternary ring in the isotopism class $[X](P, \Delta^w)$. Notice Δ^w is the image of Δ under a map fixing $p(\infty)$, $p(0)$, $l(\infty)$, and $l(0,0)$, and shifting $p(0,0)$ and $l(0)$ to the point $p(0,x)$ on $l(0,0)$ and the line $l(x)$ through $p(\infty)$, respectively. Observe also that by 4.3(1):

7.1. *Let $\bar{X} = (\bar{R}, \bar{T})$ be a ternary ring and $\bar{P} = \mathcal{P}(\bar{X})$ with basic triangle $\bar{\Delta}$. Let $w \in R^\#$. Then all shifts of R to w are isotopic, and $\bar{X}$ is a shift of R to w iff there exists an isomorphism $\varphi : \bar{P} \to P$ with $\bar{\Delta}\varphi = \Delta^w$.*

Definition 7.2. For $w \in R^\#$ let $\Lambda(w)$ consist of the triples $\lambda = (\alpha, \beta, \gamma)$ such that $\alpha, \gamma \in GL(R, +)$, $\beta \in Sym(R)$ with $0\beta = w$, and $R.(1\beta) - R.(w) = \alpha^{-1}\gamma$, where by convention $R.(0) = 0$.

7.3.

(1) *For each pair of distinct $a, b \in R$, $R.(a) - R.(b) \in GL(R, +)$.*
(2) *Let $w \in R^\#$ and $\beta \in Sym(R)$ with $0\beta = w$ and $1\beta = 0$. Then $(-R.(w)^{-1}, \beta, 1) \in \Lambda(w)$.*

Proof. Choose a, b as in (1). Then by 2.5 in [A1], $R.(a) \in GL(R,+)$ if $a \in R^\#$, and if $b \in R^\#$ then:

$$\text{for } x \in R^\#, \quad xR.(a) \neq xR.(b). \tag{$*$}$$

Thus if $a = 0$ then $\theta = R.(a) - R.(b) = -R.(b) \in GL(R,+)$, so we may assume $a \neq 0$, and similarly assume $b \neq 0$. Then by $(*)$, $\ker(\theta) = 0$, completing the proof of (1).

Choose w and β as in (2) and let $\alpha = -R_*(w)^{-1}$, $\gamma = 1$, and $\lambda = (\alpha, \beta, \gamma)$. Then by construction, $R.(1\beta) - R.(w) = -R.(w) = \alpha^{-1}\gamma$. From the discussion above, $\alpha \in GL(R,+)$, so (2) holds. $\square$

Definition 7.4. Given a binary operation $\star$ on $R^\#$, let $\mathcal{R}(\star) = (R,+,\star)$ and $\mathcal{P}(\star) = \mathcal{P}(\mathcal{R}(\star))$. For $w \in R^\#$ and $\lambda = (\alpha, \beta, \gamma) \in \Lambda(w)$, define $*_\lambda$ to be the binary operation on R such that for $a \in R^\#$,

$$R_{*_\lambda}(a) = \alpha\big(R.(a\beta) - R.(w)\big)\gamma^{-1}.$$

Define $\varphi_\lambda : \bar{P} = \mathcal{P}(*_\lambda) \to P$ to map $\bar{p}(\infty)$ to $p(\infty)$, $\bar{l}(\infty)$ to $l(\infty)$, and for $x, y \in R$, map $\bar{p}(x)$ to $p(x\alpha)$, $\bar{p}(x,y)$ to $p(x\beta, y\gamma)$, $\bar{l}(x)$ to $l(x\beta)$, and $\bar{l}(x,y)$ to $l(x\alpha, x\alpha \cdot w + y\gamma)$.

7.5. *Let $w \in R^\#$, $\lambda \in \Lambda(w)$, and $* = *_\lambda$. Then*

(1) *$\mathcal{R}(*)$ is a right quasifield with multiplicative identity* 1.
(2) *$\varphi_\lambda : \mathcal{P}(*) \to P$ is an isomorphism with $\bar{\Delta}\varphi_\lambda = \Delta^w$, where $\bar{\Delta}$ is the basic triangle of $\mathcal{P}(*)$ used to coordinatize via $\mathcal{R}(*)$.*

Proof. Let $\varphi = \varphi_\lambda$ and $\bar{P} = \mathcal{P}(*)$. Visibly φ is a bijection with $\bar{\Delta}\varphi = \Delta^w$, so to prove (2) it remains to show:

$$\text{for all lines } l \text{ in } \bar{P}, \quad \varphi\big(\bar{P}(l)\big) = P\big(\varphi(l)\big). \tag{$*$}$$

It is straightforward to check that $(*)$ is satisfied when l is $\bar{l}(\infty)$ or $\bar{l}(x)$ for $x \in R$. Further for $x, y \in R$, $\bar{P}(\bar{l}(x,y)) = \{\bar{p}(x)\} \cup \{\bar{p}(a,b): x * a + b = y\}$, while $\varphi(\bar{l}(x,y)) = l(x\alpha, x\alpha \cdot w + y\gamma)$, which is incident with $p(x\alpha)$ and $p(u,v)$ such that $x\alpha \cdot u + v = x\alpha \cdot w + y\gamma$. Also $\varphi(\bar{p}(x)) = p(x\alpha)$ and $\varphi(\bar{p}(a,b)) = p(a\beta, b\gamma)$. Thus it remains to show that if $x * a + b = y$, then also $x\alpha \cdot a\beta + b\gamma = x\alpha \cdot w + y\gamma$.

As $* = *_\lambda$, $R_*(a) = \alpha(R.(a\beta) - R.(w))\gamma^{-1}$, so $\alpha R.(a\beta) = \alpha R.(w) + r_*(a)\gamma$. Therefore as $\gamma \in GL(R,+)$,

$$\begin{aligned} x\alpha \cdot a\beta + b\gamma &= x\alpha R.(a\beta) + b\gamma = x\alpha R.(w) + xR_*(a)\gamma + b\gamma \\ &= x\alpha \cdot w + (x * a)\gamma + b\gamma = x\alpha \cdot w = (x * a + b)\gamma = x\alpha \cdot w + y\gamma, \end{aligned}$$

completing the proof of (2).

As $\varphi : \bar{P} \to P$ is an isomorphism, $\bar{R}$ is a planer loop ring. As $0\beta = w$, $a\beta \neq w$ for $a \in R^\#$, so by 7.3, $\theta(a) = R.(a\beta) - R.(w) \in GL(R,+)$. Then as $\alpha, \gamma \in GL(R,+)$, also $R_*(a) = \alpha\theta(a)\gamma^{-1} \in GL(R,+)$. Therefore by 2.3 in [A1], $\bar{R}$ is right distributive, so as $(\bar{R},=) = (R,+)$ is a group, $\bar{R}$ is a right quasifield. Finally $R_*(1) = \alpha\theta(1)\gamma^{-1} = 1$, since $\theta(1) = \alpha^{-1}\gamma$ as $\lambda \in \Lambda(w)$. Therefore 1 is the multiplicative identity of $\bar{R}$, completing the proof of (1). $\square$

7.6. *Let $w \in R^{\#}$. Then*

(1) *The shifts of R to w are of the form $(R', T_{R'})$, where R' is a right quasifield isotopic to $\bar{R} = (R, +, *)$, and for some (fixed) permutation β of R with $0\beta = w$ and $1\beta = 0$, and for each $a \in R^{\#}$,*

$$R_*(a) = 1 - R.(w)^{-1} R.(a\beta).$$

(2) *$\mathcal{H}(\bar{R}) = (R, 0, 1, K, K_{\bar{L}})$, where $K_{\bar{L}} = \{1\} \cup \{1 - R.(w)^{-1}k\colon\ K \in K_L - \{R.(w)\}\}$.*

Proof. By 7.1 all shifts are isotopic and $X' = (R', T')$ is in the set $\mathcal{X} = [X](P, \Delta^w)$ of shifts to w iff there is an isomorphism $\varphi : \mathcal{P}(X') \to P$ with $\Delta'\varphi = \Delta^w$, where Δ' is the basic triangle used to coordinatize $\mathcal{P}(X')$ via X'. By definition, $* = *_\lambda$ and $\bar{R} = \mathcal{R}(*)$, where λ is the member of $\Lambda(w)$ defined in 7.3(2). Thus by 7.5, $\bar{R}$ is a right quasifield, and $\bar{X} = (\bar{R}, T_{\bar{R}}) \in \mathcal{X}$. Then by 5.6, if $X' \in \mathcal{X}$ then $T' = T_{R'}$ and R' is a right quasifield, completing the proof of (1). Then (2) follows from the definition of $*$ in (1). $\square$

8. Geotopy of quasifields

In this section assume $R = (R, +, \cdot)$ is a right quasifield, and let $P = \mathcal{P}(R)$ with basic triangle $\Delta = (p(0,0), p(0), p(\infty))$.

Let $\simeq$ be the transitive extension of the reflection and shift relations on isotopy classes of quasifields.

8.1.

(1) *The reflection and shift relations are symmetric.*
(2) *$\simeq$ is an equivalence relation.*
(3) *If $\varphi : \bar{P} \to P$ is an isomorphism defining a shift as in 7.1, then $\varphi(\bar{p}(\infty)) = p(\infty)$ and $\varphi(\bar{l}(0,0)) = l(0,0)$.*

Proof. From 6.1 and 7.1, reflections and shifts are defined by isomorphisms $\varphi : \bar{P} \to P$ with $\bar{\Delta}\varphi = \Delta'$ or Δ^w for some $w \in R^{\#}$, respectively. Observe that also $\varphi^{-1} : P \to \bar{P}$ defines a reflection or shift from $\bar{R}$ to R in the respective case. In the case of a reflection, this is clear. Suppose φ defines a shift. Then φ maps $\bar{p}(\infty)$ to $p(\infty)$ and $\bar{l}(0,0) = \bar{p}(0) + \bar{p}(0,0)$ to $p(0) + p(0,w) = l(0,0)$, and hence $\bar{l}(0,0) = l(0,0)\varphi^{-1}$. Note the first remark establishes (3), while by the second, φ^{-1} maps $p(0,0)$ to some point of $\bar{l}(0,0)$ distinct from $\bar{p}(0)$ and $\bar{p}(0,0)$, so $p(0,0)\varphi^{-1} = \bar{p}(0,u)$ for some $u \in \bar{R}^{\#}$, completing the proof of the observation.

Of course the observation establishes (1), and (1) implies (2). $\square$

8.2. *Let $\bar{R}$ be a right quasifield with basic triangle $\bar{\Delta}$ in $\bar{P} = \mathcal{P}(\bar{R})$. Then the following are equivalent:*

(1) *R is geotopic to $\bar{R}$.*
(2) *$R \simeq \bar{R}$.*
(3) *There exists an isomorphism $\varphi : \bar{P} \to P$ with $\bar{p}(\infty)\varphi = p(\infty)$ and $\bar{l}(0,0)\varphi = l(0,0)$.*

Proof. Trivially (3) implies (1). We next show that (2) implies (3), so assume that (2) holds. Then there exists a sequence $\bar{R} = R_0, \ldots, R_n = R$ of quasifields such that R_{i+1} is a reflection or shift of R_i for $0 \leqslant i < n$. Moreover, it suffices to show there is an isomorphism $\varphi : \mathcal{P}(R_i) \to \mathcal{P}(R_{i+1})$ with $p(\infty)\varphi_i = p(\infty)$ and $l(0,0)\varphi_i = l(0,0)$. Thus we may assume $\bar{R}$ is a reflection or shift of R. Now 8.1(3) completes the proof that (2) implies (3).

Finally assume $\bar{R}$ is geotopic to R; we must show (2) holds. Then there exists an isomorphism $\varphi : \bar{P} = \mathcal{P}(\bar{R}) \to P$. If R is a field then P is Desarguesian then $A = Aut(P)$ is transitive on triangles of P, so for some $\alpha \in A$, $\varphi\alpha$ defines a reflection of R to $\bar{R}$, and hence (2) holds. Thus we may assume R is not a field, so by 5.5, $\bar{p}(\infty)\varphi = p(\infty)$. By 5.4, A is transitive on the lines of P not incident with $p(\infty)$. So again replacing φ by $\varphi\beta$ for suitable $\beta \in A$, we may assume $\bar{l}(0,0)\varphi = l(0,0)$. That is (1) implies (3).

Next $\bar{p}(0)$ is incident with $\bar{l}(0,0)$, so $\bar{p}(0)\varphi$ is incident with $l(0,0)$, and hence $\bar{p}(0)\varphi = p(0)$ or $p(0,u)$ for some $u \in R$. Suppose $\bar{p}(0)\varphi = p(0)$. Then $\bar{p}(0,0)\varphi \neq p(0)$ is on $l(0,0)$, so $\bar{p}(0,0) = p(0,w)$ for some $w \in R$. If $w = 0$ then $\bar{R}$ is isotopic to R by 4.3, so $R \simeq \bar{R}$, while if $w \neq 0$ then $\bar{R}$ is a shift of R to w, and again $R \simeq \bar{R}$.

Thus we may assume $\bar{p}(0)\varphi = p(0,u)$ for some $u \in R$. Let $X_1 = (R_1, T_1) \in [X](P, \Delta^u)$. Then X_1 is a shift of R to u, so $R \simeq R_1$, and by 7.6, $T_1 = T_{R_1}$ and R_1 is a right quasifield. Thus it suffices to show that $\bar{R} \simeq R_1$, so replacing R by R_1, we may assume $u = 0$.

Similarly $\bar{p}(0,0)\varphi = p(0)$ or $p(0,v)$. In the former case, φ defines a reflection of R to $\bar{R}$, so $R \simeq \bar{R}$; thus we may assume the latter case holds. Now define Δ' as in Section 6, and let $X_2 = (R_2, T_2) \in [X](P, \Delta')$. Then X_2 is a reflection of R, so as above, we may replace R by R_2 we now have φ mapping $\bar{p}(0)$ to $p(0)$ and $\bar{p}(0,0)$ to $p(0,v)$, so that $\bar{R}$ is a shift of R to v, completing the proof. □

9. Homologies

Notation 9.1. Recall the *middle nucleus* $Nuc_m(Y)$ of a loop $(Y, \circ)$ is the set of $y \in Y$ such that

$$(x \circ y) \circ z = x \circ (y \circ z) \qquad (*)$$

for all $x, z \in Y$. Set $N_m(R) = Nuc_m(R^\#, \cdot)$. Similarly the *right nucleus* $Nuc_r(Y)$ consists of those $z \in Y$ such that $(*)$ holds for all $x, y \in Y$, and we set $N_r(R) = Nuc_r(R^\#, \cdot)$.

For $R \in \mathbf{R}$ and $1 \leqslant i \leqslant 3$, write $\mathrm{top}_i(R)$ for the set of $(\alpha_1, \alpha_2, \alpha_3) \in \mathrm{aut}(R)$ such that $\alpha_i = 1$. Write $\mathrm{prin}(R)$ for the group $\mathrm{top}_3(R)$ of principal autotopisms R.

Remark 9.2. We recall some more facts from Chapter IV of [HP]: Let (p, l) be a *nonincident* point–line pair in a projective plane P. A *homology* with *center* p and *axis* l is an automorphism γ of P with fixes each point on l and each line through p. Again if γ is a nontrivial homology then the fixed elements of γ consist of p, l, and the elements incident with p or l.

Write $\mathbf{H}(p, l)$ for the set of all homologies with center p and axis l, and observe $H = \mathbf{H}(p, l)$ is a group semiregular on $P(m) - \{p, m \cap l\}$ for each $m \in P(p)$, and dually. The plane P is (p, l)-transitive if H is transitive on all these sets, or equivalently H is regular on each set; if $P = \mathcal{P}(R)$ for some finite $R \in \mathbf{R}$, then equivalently $|H| = |R^\#|$.

The two lemmas in this section appear to be well known, at least for certain classes of rings.

9.3. *Let $R \in \mathbf{R}$. Then*

(1) *The map $x \mapsto R(x)$, $x \mapsto L(x)$, is an injective loop homomorphism, loop antihomorphism, from $N_m(R)$ into the right, left multiplication group of $(R^{\#}, \circ)$, respectively. In particular, $N_m(R)$ is a group.*
(2) *The map $\eta : u \mapsto (R(u), L(u^{-1}), 1)$ is an isomorphism of $N_m(R)$ with* $\mathrm{prin}(R)$.
(3) *For $u \in N_m(R)$, $\partial(u) = \mathcal{P}(\eta(u))$ is a homology of $P = \mathcal{P}(R)$ with center $p(0)$ and axis $l(0)$. Further $\partial(u)$ maps $l(x)$ to $l(u^{-1} \cdot x)$, $p(x)$ to $p(x \cdot u)$, $l(x, y)$ to $l(x \cdot u, y)$, and $p(x, y)$ to $p(u^{-1} \cdot x, y)$.*
(4) *Suppose $\psi = (\alpha, \beta, \gamma) : R \to R'$ is an isotopism. Then $\psi^* : \eta(u) \mapsto \eta(u)^{\psi} = (R(u)^{\alpha}, L(u^{-1})^{\beta}, 1)$ is an isomorphism of* $\mathrm{prin}(R)$ *with* $\mathrm{prin}(R')$, *such that $R(u)^{\alpha} = R'(uL(1'\alpha^{-1})\alpha)$. Therefore $\psi_\eta = \eta\psi^*\eta'^{-1} : u \mapsto uL(1'\alpha^{-1})\alpha$ is an isomorphism of $N_m(R)$ with $N_m(R')$.*
(5) *If $R \in \mathbf{R}^*$ then $\partial : N_m(R) \to$* $\mathrm{aut}(\mathcal{P}(R))$ *induces an isomorphism of $N_m(R)$ with $\mathbf{H}(p(0), l(0))$.*

Proof. Parts (1) and (2) are well known and easy. By 3.3(1), $\partial(u)$ is an automorphism of P, and by inspection it acts as indicated in (3).

Choose ψ as in (4). Then by general nonsense, $\psi^* : \mathrm{prin}(R) \to \mathrm{prin}(R')$ is an isomorphism. By (2), there exists $u_* \in N_m(R')$ with $(R(u)^{\alpha}, L(u^{-1})^{\beta}, 1) = (R(u_*), L(u_*^{-1}), 1)$. Let $1'$ denote the multiplicative identity of R'. Then

$$u_* = 1'R'(u_*) = 1'R(u)^{\alpha},$$

so $u_*\alpha^{-1} = 1'\alpha^{-1} \circ u = uL(1'\alpha^{-1})$, completing the proof of (4).

Let $H = \mathbf{H}(p(0), l(0))$. By 3.5(2), $\mathcal{P} : \mathrm{top}(R) \to \mathrm{aut}(P)$ is an isomorphism, so $\mathcal{P}$ induces an isomorphism $J \to H$, where $J = \mathcal{P}^{-1}(H) \leqslant \mathrm{top}(R)$. If $\psi = (\alpha, \beta, \gamma) \in J$, then $\mathcal{P}(\psi)$ fixes $l(0, y)$ for each $y \in R$, so as $l(0, y)\mathcal{P}(\psi) = l(0, y\gamma)$, $\gamma = 1$. That is $\psi \in \mathrm{prin}(R)$, so $J \leqslant \mathrm{prin}(R)$. By (2), $\eta(N(R)) = \mathrm{prin}(R)$, so by (3), $\mathrm{prin}(R) \leqslant J$. Then (2) completes the proof of (5). □

9.4. *Let $R \in \mathbf{R}$. Then*

(1) *The map $x \mapsto R(x)$ is an injective loop homomorphism from $N_r(R)$ into the enveloping group of $(R^{\#}, \circ)$. In particular, $N_r(R)$ is a group.*
(2) *The map $\zeta : u \mapsto (1, R(u), R(u))$ is an isomorphism of $N_r(R)$ with* $\mathrm{top}_1(R)$.
(3) *For $u \in N_r(R)$, $\rho(u) = \mathcal{P}(\zeta(u))$ is a homology of $P = \mathcal{P}(R)$ with center $p(0, 0)$ and axis $l(\infty)$. Further for $x, y \in R$, $\rho(u)$ fixes $p(\infty)$, $l(\infty)$, and $p(x)$, while $\rho(u)$ maps $l(x)$ to $l(xu)$, $l(x, y)$ to $l(x, yu)$, and $p(x, y)$ to $p(xu, yu)$.*
(4) *If $R \in \mathbf{R}^*$ then $\rho : N_r(R) \to$* $\mathrm{aut}(\mathcal{P}(R))$ *induces an isomorphism of $N_r(R)$ with $\mathbf{H}(p(0, 0), l(\infty))$.*

Proof. The proof is much like that of the previous lemma. Again parts (1) and (2) are well known and easy; for example, (2) is 5.4 in [Ki]. By 3.3(1), $\rho(u)$ is an automorphism of P, and by inspection it acts as indicated in (3). The proof (4) is essentially the same as the proof of 9.3(5). □

10. Isotopy and geotopy in $\mathbf{R}^!$

Recall $\mathbf{T}^*$ is the subcategory of $\mathbf{T}$ consisting of the planar ternary rings, and $\mathbf{R}^*$ is the subcategory of $\mathbf{R}$ consisting of the planar loop rings. Write $\mathbf{R}^!$, $\mathbf{R}^{!d}$ for the subcategories of $\mathbf{R}^*$ consisting of those finite rings R such that the multiplicative loop $(R^\#, \cdot)$ of R is a group, and such that R is right, left distributive, respectively. Write $\mathbf{T}^!$, $\mathbf{T}^{!d}$ for the linear ternary rings (R, T_R) with $R \in \mathbf{R}^!$, $\mathbf{R}^{!d}$, respectively.

In this section we focus on planes $\mathcal{P}(R)$ for $R \in \mathbf{R}^!$.

10.1. $\mathbf{R}^{!d}$ *consists of the rings* R^{op}*,* $R \in \mathbf{R}^!$.

Proof. This follows from 4.8(2). $\square$

10.2. *Let* $X = (R, T) \in \mathbf{T}^*$*. Then* $\mathcal{P}(X)$ *is* $(p(0), l(0))$*-transitive iff* T *is linear and* $(R^\#, \cdot)$ *is a group.*

Proof. This is Theorem 6.5 in [HP]. $\square$

10.3. *Let* $R \in \mathbf{R}^*$*. Then*

(1) $\mathcal{P}(R)$ *is* $(p(\infty), l(0,0))$*-transitive iff* $R \in \mathbf{R}^{!d}$.
(2) $\mathcal{P}(R)$ *is* $(p(0,0), l(\infty))$*-transitive iff* $R \in \mathbf{R}^!$.

Proof. Part (1) is Theorem 6.6 in [HP], while (2) is the dual of (1); dualize using 4.8. $\square$

10.4. *Let* $R \in \mathbf{R}^!$ *be of order* n*,* $P = \mathcal{P}(R)$*,* $\Delta = (p(0,0), p(0), p(\infty))$*, and* L *the multiplicative group of* R*. Then*

(1) $L = N_m(R) = N_r(R)$.
(2) *The map* $\rho : L \to \mathbf{H}(p(0,0), l(\infty))$ *of* 9.4(4) *is an isomorphism, and* P *is* $(p(0,0), l(\infty))$*-transitive.*
(3) *The map* $\partial : L \to \mathbf{H}(p(0), l(0))$ *of* 9.3(5) *is an isomorphism, and* P *is* $(p(0), l(0))$*-transitive.*
(4) *Let* $L^2 = L \times L$ *and define* $\partial \times \rho : L^2 \to Aut(P)$ *by* $(\partial \times \rho)(u, v) = \partial(u)\rho(v)$*. Then* $\partial \times \rho$ *is an injective group homomorphism,* $\partial(u)\rho(v)$ *fixes* $p(\infty)$ *and* $l(\infty)$*, maps* $l(x)$ *to* $l(u^{-1}xv)$*,* $p(x)$ *to* $p(xu)$*,* $l(x, y)$ *to* $l(xu, yv)$*, and* $p(x, y)$ *to* $p(u^{-1}xv, yv)$.
(5) $(\partial \times \rho)(L^2)$ *is regular on* $\Gamma(\Delta)$ *and on the set of lines through no point of* Δ*. Further* $(\partial \times \rho)(L^2)$ *is transitive on* $P(l) - \Delta$ *and on* $P(p) - \Delta'$ *for each* l *in the set* Δ' *of lines through two points of* Δ*, and each* $p \in \Delta$.

Proof. As L is a group, (1) holds. Then 9.4 implies (2) and 9.3 implies (3). By (2) and (3), $\gamma = \partial \times \rho$ maps L into $Aut(P)$, and a straightforward calculation using the descriptions of ∂ and ρ in 9.3 and 9.4, shows that γ is as claimed in (4). Then visibly γ is an injective group homomorphism, completing the proof of (4). Part (5) follows from inspecting (4). $\square$

10.5. *Let* $X = (R, T) \in \mathbf{T}^*$*,* $P = \mathcal{P}(X)$*, and* $\Delta = (p(0,0), p(0), p(\infty))$*. Then the following are equivalent*:

(1) $X \in \mathbf{T}^!$.
(2) *Each member of* $[X](P, \Delta)$ *is in* $\mathbf{T}^!$.
(3) P *is* $(p(0), l(0))$ *and* $(p(0,0), l(\infty))$*-transitive.*
(4) P^* *is* $(p(0), l(0))$ *and* $(p(\infty), l(0,0))$*-transitive.*
(5) *Let* $\Delta^* = (p^*(0,0), p^*(0), p^*(\infty))$. *Each member of* $[X](P^*, \Delta^*)$ *is in* $\mathbf{T}^{!d}$.

Proof. By 10.2 and 10.3, (3) is equivalent to each of (1) and (2). From the definition of P^*, $Aut(P) = Aut(P^*)$, and (3) and (4) are equivalent. Then by another application of 10.2 and 10.3, (4) and (5) are equivalent. □

10.6. *Assume* $R \in \mathbf{R}^!$ *and let* L *be the multiplicative group of* R, $P = \mathcal{P}(R)$, $\Delta = (p(0,0), p(0), p(\infty))$, $A = Aut(P)$, $B = A_\Delta$, *and* $M = (\partial \times \rho)(L \times L)$. *Let* $I = \{p(0,0), p(0), p(\infty)\}$, *and represent* $S = Sym(I)$ *on the set* Σ *of triangles of* 4.7 *as in that lemma. Let*

$$S^+ = \{s \in S\colon s \text{ extends to an automorphim of } P\} \quad \text{and} \quad S^- = \langle t_0 \rangle.$$

Let $t_0 = (p(0,0), p(0))$ *and* $\tau = (p(0,0), p(\infty))$ *be transpositions in* S. *Then*

(1) *For each* $s \in S$, $[X](P, \Delta s)$ *is an isomorphism class of planer ternary rings* $X_s = (R_s, T_s)$.
(2) $B \cong \mathrm{top}(X_s)$ *for each* $s \in S$.
(3) $B_{p(1,1)} \cong Aut(X_s)$ *for each* $s \in S$.
(4) B *is the semidirect product of* $M = (\partial \times \rho)(L \times L) \cong L \times L$ *by* $B_{p(1,1)} \cong Aut(R)$.
(5) $X_{t_0} \in \mathbf{T}^!$.
(6) *For* $s \in \tau S^-$, $X_s \in \mathbf{T}^{!d}$.
(7) *If* L *is abelian, then for all* $s \in S$, $X_s \in \mathbf{T}^!$, R_s *is distributive, and* L *is isomorphic to the multiplicative group of* R_s.
(8) *If* L *is nonabelian then*:
   (a) P *is not* $(p(\infty), l(0,0))$*-transitive.*
   (b) R_s *is not distributive for* $s \in S^- \cup \tau S^-$.
   (c) R_s *is not linear for* $s \in t\tau S^-$.
   (d) $S^+ = 1$ *or* S^-.
   (e) *The multiplicative loop of* R_s *is isomorphic to* L^{op} *for* $s \in \tau S^-$.

Proof. By 10.4(5), M is regular on $\Gamma = \Gamma(\Delta)$, so by 4.5(3), $[X](P, \Delta_s) = [X_s]$ is the isomorphism class of some ternary ring $X_s = (R_s, T_s)$. By 10.5, $T_s = T_{R_s}$ and $R_s \in \mathbf{R}^!$, establishing (1). Then 4.6(1) implies (2) and 4.6(4) implies (3).

As M is regular on Γ, $B_{p(1,1)}$ is a complement to M in B. From the description in 4.3(4), $B_{p(1,1)}$ acts on M, so the product is semidirect, and by 10.4(4), $M \cong L \times L$, so (4) holds. By 10.5, P is $(p(0), l(0))$ and $(p(0,0), l(\infty))$-transitive. Thus $\mathcal{P}(X_{t_0})$ also has this symmetry, so (5) follows from another application of 10.5. Similarly for $s \in \tau S^-$, $\mathcal{P}(X_s)$ is $(p(0), l(0))$ and $(p(\infty), l(0,0))$-transitive, so (6) follows from the dual of 10.5.

Assume L is abelian. Then as R is right distributive, R is distributive so $P \in \mathbf{T}^{!d}$. Thus P is $(p(\infty), l(0,0))$-transitive by 10.3, so P is (p, l)-transitive for each $p \in I$ and line l opposite to p in the triangle Δ. Then (7) follows from previous parts of the lemma applied to each X_s, except for the statement that L is the multiplicative group L_s of X_s for each $s \in S$, which we now prove: Namely by 10.4, L_s isomorphic to two of the three homology groups at the three points of Δ, so $L \cong L_s$. Note the same argument gives part (e) of (8).

Finally assume L is nonabelian. Then by a result of Kantor and Pankin in [KP], R is *not* distributive, so P is not $(p(\infty), l(0,0))$-transitive. Thus (8)(a) holds, and (8)(a) and 10.5 imply (8)(b) and (8)(d).

It remains to prove (8)(c), so we may assume X_s is linear for some $s \in t_0\tau S^-$, and it remains to derive a contradiction. Now $\mathcal{P}(X_s)$ is $(p(0,0), l(\infty))$ but not $(p(0), l(0))$-transitive, so by 10.3, $R_s \in \mathbf{R}^!$, so $\mathcal{P}(X_s)$ *is* $(p(0), l(0))$-transitive by 10.5, a contradiction. □

10.7. *Assume R is a finite right nearfield which is not a field and adopt the notation of Lemma* 10.6. *Then*

(1) $X = X_1 \cong X_{t_0}$.
(2) $X_\tau \in \mathbf{T}^{!d}$, *$R_\tau$ is not distributive, and* $X_\tau \cong X_{\tau t_0}$.
(3) *$X_{t_0\tau}$ is not linear and* $X_{t_0\tau} \cong X_{t_0\tau t_0}$.
(4) *The additive loops of R_τ and $R_{t\tau}$ are not associative.*
(5) $S^+ = \langle t_0 \rangle$.
(6) *$R_\tau^{\mathrm{op}} \in \mathbf{R}^!$, but R_τ^{op} is not a right quasifield. However R_τ^{op} coordinatizes the translation plane $\mathcal{P}(R_\tau^{\mathrm{op}}) = P^*$, although not via the standard coordinatization.*
(7) *R_τ^{op} is of characteristic* $p = \mathrm{char}(R)$.

Proof. Part (1) follows from 6.3(4). By (1) and 4.7(2), $t_0 \in S^+$, so (5) follows from 10.6(8)(d), which also supplies the isomorphisms in (2) and (3). Next parts (6) and (8)(b) of 10.6 complete the proof of (2), while 10.6(8)(c) completes the proof of (3).

If (4) fails then R_τ coordinatizes P as $\mathcal{P}(R_\tau)$, the plane of the left nearfield R_τ with axis $l = l(0,0)$. So by the dual of 5.4, the group $E(l)$ of elations of P with axis l is transitive on the set Θ of points of P not on l. This contradicts 5.5, which says $Aut(P)$ fixes $p(\infty)$, whereas $p(\infty) \in \Theta$. So (4) holds.

Part (6) follows from (2), (4), 5.1, and 4.8.

As R is a right near field, R is of characteristic p for some prime p, and hence (cf. 1.4 in [A2]) the subring $\bar{R}$ of R generated by 1 is the field $\mathbf{F}_p$ of order p and by 4.9(5),

$$\bar{P} = \{p(\infty), l(\infty), p(x), l(x), p(x,y), l(x,y)\colon x, y \in R\}$$

is a subplane of P isomorphic to the Desarguesian plane $\mathcal{P}(\bar{R})$. Let $R' = (\bar{R}, +', \cdot')$ be $\bar{R}$ under the addition $+'$ and multiplication $\cdot'$ in R_τ^{op}. Then the dual $\bar{P}^*$ of $\bar{P}$ in $P^* \cong \mathcal{P}(R_\tau^{\mathrm{op}})$ is Desarguesian of order p, so (7) follows from 4.9(4). □

10.8. *Assume $R, \bar{R} \in \mathbf{R}^!$ and $\varphi : \mathcal{P}(R) \to \mathcal{P}(\bar{R})$ is an isomorphism. Assume further the multiplicative group of R is nonabelian. Then*

(1) *There exists $\alpha \in Aut(\mathcal{P}(R))$ such that $(p(0,0), p(0), p(\infty))\alpha\varphi = (p(0,0), p(0), p(\infty))$ or $(p(0), p(0,0), p(\infty))$, and $R \cong \bar{R}$ or $R_{t_0} \cong \bar{R}$, where $t_0 = (p(0), p(0,0))$.*
(2) *Either $\bar{R}$ is isotopic to R, or $\bar{R}$ is a reflection of R.*

Proof. Adopt the notation of 10.6 and the corresponding notation for $\bar{R}$. Let $\mathcal{D}$ be the set of triangles Δ' in P such that $[X](P, \Delta') = \{X'\}$ with $X' = (R', T') \in \mathbf{R}^!$.

Identifying P with $\bar{P}$ via φ, we may assume without loss of generality that $P = \bar{P}$ and $\varphi = 1$. Thus $\Delta, \bar{\Delta} \in \mathcal{D}$.

Let q be a prime divisor of $|L|$ and pick q odd if possible. Let $M \leqslant U \leqslant V$ with $U \in Syl_q(B)$ and $V \in Syl_q(A)$. Similarly $\bar{M} \leqslant \bar{U} \leqslant \bar{V}$. By 10.4, $I = Fix(M)$ is the set of fixed points of M in its action on the points of P, so also $I = Fix(U)$. Then as $N_V(U)$ acts on I and $|Aut_A(I)| \leqslant 2$ by 10.6(8)(d), either

(a) $U = V$, or
(b) $q = 2$ and $|N_V(U) : U| = 2$.

Suppose for the moment that case (b) holds. Then by choice of q, L is a 2-group. Thus L' is a 2-group for each $\Delta' \in \mathcal{D}$, so by Proposition 1 in [A2], L' has a unique involution, and hence L' is quaternion or cyclic. Let $\mathcal{M}$ be the set of subgroups M' of $N_V(U)$ such that $M' = (\partial)((L')^2)$ for some $\Delta' \in \mathcal{D}$.

Let $M' \in \mathcal{M}$. The index of $M'_U = M' \cap U$ in M' is at most $|N_V(U) : U| = 2$, so $\Omega_1(M') = \Omega_1(M'_U) \cong E_4$. Further from 10.4 and as $M' \in \mathcal{M}$, $Fix(\Omega_1(M')) = I'$, so $I = I'$ and $M' \leqslant U$. Also $M' = \mathbf{H}(p_1, l_1) \times \mathbf{H}(p_2, l_2)$ for some $p_i \in I'$ and l_i the line in Δ' determined by I' by 10.4. It follows from 10.6(8)(a) that $\{p_1, p_2\} = \{p(0,0), p(0)\}$, and then that $M' = M$.

We have shown that either $V = U$ or (b) holds and $\mathcal{M} = \{M\}$. In the latter case, M is normal in $N_V(N_V(U))$, so $N_V(N_V(U))$ acts on $Fix(M) = I$, and hence $N_V(N_V(U)) = N_V(U)$, so that $V = N_V(U)$ and $|V : U| = 2$.

By Sylow's theorem there is $\alpha \in A$ with $\bar{V}^\alpha = V$. Thus in (a), $\bar{I}\alpha = Fix(\bar{V}^\alpha) = Fix(V) = I$, while in (b), $\bar{M}^\alpha = M$ as $\mathcal{M} = \{M\}$, so $\bar{I}\alpha = Fix(\bar{M}^\alpha) = Fix(M) = I$. Further if $\bar{\Delta} = (p_1, p_2, p_3)$ then by 10.6(8)(a), $\{p_1, p_2\} = \{p(0,0), p(0)\}$, and then (1) follows from 10.6(1).

Notice (1) and 6.1 imply (2). □

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